

CPA

Certified Public Accountant Examination

Stage: Foundation F1.1

Subject Title: Business Mathematics &
Quantitative Methods

Study Manual



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**INSTITUTE OF
CERTIFIED PUBLIC ACCOUNTANTS
OF
RWANDA
Foundation F1**

**F1.1 BUSINESS MATHEMATICS &
QUANTITATIVE METHODS**

First Edition 2012

This study manual has been fully revised and updated
in accordance with the current syllabus.

It has been developed in consultation with experienced lecturers.

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Stage: Foundation 1

Subject Title: F1.1 Business Mathematics and Quantitative Methods

Aim

The aim of this subject is to ensure that students acquire, understand and apply quantitative techniques that are used in business decision-making. They develop the ability to interpret the information obtained and present this information in a manner appropriate to a business environment.

Business Mathematics and Quantitative Methods as an Integral Part of the Syllabus

This is an essential foundation subject for the professional accountant. It develops the mathematical and statistical competence necessary to facilitate students' progression through the Foundation and Advanced Level examinations in subjects such as *Financial Accounting, Financial Reporting, Advanced Financial Reporting, Management Accounting, Managerial Finance, Strategic Corporate Finance and Strategic Performance Management*.

Learning Outcomes

On successful completion of this subject students should be able to:

- Demonstrate the use of financial mathematics, measures of central tendency / dispersion and indices in business.
- Display information in a graphical/tabular form including frequency distributions, networks, etc.
- Demonstrate the use of probability and confidence intervals in business.
- Explain the concept of present value and apply discounting techniques in investment appraisal.
- Apply moving averages and regression analysis in forecasting.

Syllabus:

1. Introduction to Financial Mathematics

- Simple and compound interest, annual percentage rate, (APR), depreciation, (straight line and reducing balance), discounting, present and future value of money and investment appraisal techniques, annuities, mortgages, amortisation, sinking funds.
- Handling formulae, use of positive and negative numbers, brackets and powers, calculus.
- Linear and quadratic equations and graphs: costs and production functions (fixed, variable and total costs, average and marginal costs): break-even analysis, revenue and profit functions and their interpretation.

2. Sources Of Data, Presentation And Use

- Sources and types of data (primary & secondary data), nature, appreciation and precautions in use.
- Role of statistics, uses and misuses of statistics in business analysis and decision making
- Presentation of data, use of bar charts, histograms, pie charts, graphs, tables, frequency distributions, histogram, frequency polygons, ogives and their use and interpretation

3. Measures of Central Tendency and Dispersion

- Averages and variations for grouped and ungrouped data
- Measures of location – mean, median, mode, geometric mean, harmonic mean, percentiles, quartiles.
- Measures of dispersion – range, variance, standard deviation, co-efficient of variation

4. Probability and Probability Distributions

- Meaning of probability, nature of probability distributions, discrete and
- Continuous random variables, expected values.
Standard Normal Distribution, confidence intervals, z-score, T-Chi square and associated diagrams
- Use and application of probability distributions.
- Binomial probability distribution and its application in business
- Analysis of binominal populations (the probability of success p, and failure q)
- The Poisson Probability Distribution, the Poisson population and application of Poisson distribution in analysing of Poisson events.

5. Sampling and Sampling Theory (The role of sampling as compared to population census)

- Probability Sampling Methods – Simple random, stratified, cluster, Systematic sampling
- Interval estimation for large and small samples; confidence levels, standard error; estimate of sample size.
- Hypothesis testing – Null and Alternative hypothesis; description of Type I and Type II errors. Non Probability sampling methods = quota sampling and snowball sampling.

6. Regression and Correlation Analysis

- Simple Linear Regression, scatter graphs, least squares method.
- Co-efficient of determination, correlation co-efficient, rank and product moment correlation.
- Use of linear regression equation in forecasting.

7. Time Series Analysis

- Factors influencing time series – trend, seasonal, cyclical, irregular variations.
- Smoothing time series by means of moving averages.
- Use of time series in forecasting

8. Indices: Use And Construction

- Simple, aggregate, Laspeyres, Paasche, chain indices.
- Change of base period, weighting.
- Construction, use and interpretation of indices.

9. Network Analysis

- Activity identification, Relationship between various elements, construction of simple networks.
- Analysis of networks by deriving the critical and non-critical activities.
- Derivation and definition of the critical path.

10. Linear Programming

- Simple Linear Programming and simplex
- Transportation
- Assignments

11. Decision Theory

- Minimax, Maximum, Maximax
- Decision Trees
- Game Theory

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STUDY UNIT 1

Probability

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A. ESTIMATING PROBABILITIES

Introduction

Suppose someone tells you “there is a 50-50 chance that we will be able to deliver your order on Friday”. This statement means something intuitively, even though when Friday arrives there are only two outcomes. Either the order will be delivered or it will not. Statements like this are trying to put probabilities or chances on uncertain events.

Probability is measured on a scale between 0 and 1. Any event which is impossible has a probability of 0, and any event which is certain to occur has a probability of 1. For example, the probability that the sun will not rise tomorrow is 0; the probability that a light bulb will fail sooner or later is 1. For uncertain events, the probability of occurrence is somewhere between 0 and 1. The 50-50 chance mentioned above is equivalent to a probability of 0.5.

Try to estimate probabilities for the following events. Remember that events which are more likely to occur than not have probabilities which are greater than 0.5, and the more certain they are the closer the probabilities are to 1. Similarly, events which are more likely not to occur have probabilities which are less than 0.5. The probabilities get closer to 0 as the events get more unlikely.

- (a) The probability that a coin will fall heads when tossed.
- (b) The probability that it will snow next Christmas.
- (c) The probability that sales for your company will reach record levels next year.
- (d) The probability that your car will not break down on your next journey.
- (e) The probability that the throw of a dice will show a six.

The probabilities are as follows:

- (a) The probability of heads is 0.5.
- (b) This probability is quite low. It is somewhere between 0 and 0.1.
- (c) You can answer this one yourself.
- (d) This depends on how frequently your car is serviced. For a reliable car it should be greater than 0.99.
- (e) The probability of a six is $1/6$ or 0.167.

Theoretical Probabilities

Sometimes probabilities can be specified by considering the physical aspects of the situation. For example, consider the tossing of a coin. What is the probability that it will fall heads? There are two sides to a coin. There is no reason to favour either side as a coin is symmetrical. Therefore the probability of heads, which we call $P(H)$ is:

$$P(H) = 0.5.$$

Another example is throwing a dice. A dice has six sides. Again, assuming it is not weighted in favour of any of the sides, there is no reason to favour one side rather than another. Therefore the probability of a six showing uppermost, $P(6)$, is:

$$P(6) = 1/6 = 0.167.$$

As a third and final example, imagine a box containing 100 beads of which 23 are black and 77 white. If we pick one bead out of the box at random (blindfold and with the box well shaken up) what is the probability that we will draw a black bead? We have 23 chances out of 100, so the probability is:

$$\frac{23}{100} \quad (\text{or } P = 0.23)$$

Probabilities of this kind, where we can assess them from our prior knowledge of the situation, are also called “a priori” probabilities.

In general terms, we can say that if an event E can happen in h ways out of a total of n possible equally likely ways, then the probability of that event occurring (called a success) is given by:

$$P(E) = \frac{h}{n}$$

$$= \frac{\text{Number of possible ways of E occurring}}{\text{Total number of possible outcomes}}$$

Empirical Probabilities

Often it is not possible to give a theoretical probability of an event. For example, what is the probability that an item on a production line will fail a quality control test? This question can be answered either by measuring the probability in a test situation (i.e. empirically) or by relying on previous results. If 100 items are taken from the production line and tested, then:

Probability of failure $P(F) = \frac{\text{Number of items which fail}}{\text{Total number of items tested}}$
 So, if 5 items actually fail the test

$$P(F) = \frac{5}{100} = 0.05.$$

Sometimes it is not possible to set up an experiment to calculate an empirical probability. For example, what are your chances of passing a particular examination? You cannot sit a series of examinations to answer this. Previous results must be used. If you have taken 12 examinations in the past, and failed only one, you might estimate:

$$\text{Probability of passing, } P(\text{Pass}) = \frac{11}{12} = 0.92$$

B. TYPES OF EVENT

There are five types of event:

- Mutually exclusive
- Non-mutually-exclusive
- Independent
- Dependent or non-independent
- Complementary.

(a) **Mutually Exclusive Events**

If two events are mutually exclusive then the occurrence of one event precludes the possibility of the other occurring. For example, the two sides of a coin are mutually exclusive since, on the throw of the coin, “heads” automatically rules out the possibility of “tails”. On the throw of a dice, a six excludes all other possibilities. In fact, all the sides of a dice are mutually exclusive; the occurrence of any one of them as the top face automatically excludes any of the others.

(b) **Non-Mutually-Exclusive Events**

These are events which can occur together. For example, in a pack of playing cards hearts and queens are non-mutually-exclusive since there is one card, the queen of hearts, which is both a heart and a queen and so satisfies both criteria for success.

(c) **Independent Events**

These are events which are not mutually exclusive and where the occurrence of one event does not affect the occurrence of the other. For example, the tossing of a coin in no way affects the result of the next toss of the coin; each toss has an independent outcome.

(d) **Dependent or Non-Independent Events**

These are situations where the outcome of one event is dependent on another event. The probability of a car owner being able to drive to work in his car is dependent on him being able to start the car. The probability of him being able to drive to work given that the car starts is a conditional probability and

$$P(\text{Drive to work}|\text{Car starts})$$

where the vertical line is a shorthand way of writing “given that”.

(e) **Complementary Events**

An event either occurs or it does not occur, i.e. we are certain that one or other of these situations holds.

For example, if we throw a dice and denote the event where a six is uppermost by A, and the event where either a one, two, three, four or five is uppermost by \bar{A} (or not A) then A and \bar{A} are complementary, i.e. they are mutually exclusive with a total probability of 1. Thus:

$$P(A) + P(\bar{A}) = 1.$$

This relationship between complementary events is useful as it is often easier to find the probability of an event not occurring than to find the probability that it does occur. Using the above formula, we can always find $P(A)$ by subtracting $P(\bar{A})$ from 1.

C. THE TWO LAWS OF PROBABILITY

Addition Law for Mutually Exclusive Events

Consider again the example of throwing a dice. You will remember that

$$P(6) = \frac{1}{6}$$

$$\text{Similarly } P(1) = \frac{1}{6}$$

$$P(2) = \frac{1}{6}$$

$$P(3) = \frac{1}{6}$$

$$P(5) = \frac{1}{6}$$

$$P(6) = \frac{1}{6}$$

What is the chance of getting 1, 2 or 3?

From the symmetry of the dice you can see that $P(1 \text{ or } 2 \text{ or } 3) = 0.5$. But also, from the equations shown above you can see that

$$P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 0.5.$$

This illustrates that

$$P(1 \text{ or } 2 \text{ or } 3) = P(1) + P(2) + P(3)$$

This result is a general one and it is called the addition law of probabilities for mutually exclusive events. It is used to calculate the probability of one of any group of mutually exclusive events. It is stated more generally as:

$$P(A \text{ or } B \text{ or } \dots \text{ or } N) = P(A) + P(B) + \dots + P(N)$$

where A, B ... N are mutually exclusive events.

Addition Law for a Complete List of Mutually Exclusive Events

(a) If all possible mutually exclusive events are listed, then it is certain that one of these outcomes will occur. For example, when the dice is tossed there must be one number showing afterwards.

$$P(1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6) = 1.$$

Using the addition law for mutually exclusive events, this can also be stated as

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1.$$

Again this is a general rule. The sum of the probabilities of a complete list of mutually exclusive events will always be 1.

Example

An urn contains 100 coloured balls. Five of these are red, seven are blue and the rest are white. One ball is to be drawn at random from the urn.

What is the probability that it will be red?

$$\text{Probability that ball is red } P(R) = \frac{5}{100} = 0.05$$

What is the probability that it will be blue?

$$\text{Probability that ball is blue } P(B) = \frac{7}{100} = 0.07$$

What is the probability that it will be red or blue?

$$P(R \text{ or } B) = P(R) + P(B) = 0.05 + 0.07 = 0.12.$$

This result uses the addition law for mutually exclusive events since a ball cannot be both blue and red.

What is the probability that it will be white?

The ball must be either red or blue or white. This is a complete list of mutually exclusive possibilities.

$$\text{Therefore } P(R) + P(B) + P(W) = 1$$

$$P(W) = 1 - P(R) - P(B)$$

$$= 1 - 0.05 - 0.07$$

$$= 0.88$$

Addition Law for Non-Mutually-Exclusive Events

Events which are non-mutually-exclusive are, by definition, capable of occurring together.

The addition law can still be used but the probability of the events occurring together must be deducted:

$$P(\text{A or B or both}) = P(\text{A}) + P(\text{B}) - P(\text{A and B}).$$

Examples

- (a) If one card is drawn from a pack of 52 playing cards, what is the probability: (i) that it is either a spade or an ace; (ii) that it is either a spade or the ace of diamonds?

- (i) Let event B be “the card is a spade”. Let event A be “the card is an ace”.

We require $P(\text{spade or ace [or both]}) = P(\text{A or B})$

$$= P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

$$P(\text{A}) = \frac{\text{No. of aces}}{\text{No. in pack}} = \frac{4}{52}$$

$$P(\text{B}) = \frac{\text{No. of spades}}{\text{No. in pack}} = \frac{13}{52}$$

$$P(\text{A and B}) = \frac{\text{No. of aces of spades}}{\text{No. in pack}} = \frac{1}{52}$$

$$\therefore P(\text{spade or ace}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

(ii) Let event B be “the card is a spade”.

Let event A be “the card is the ace of diamonds”.

$$P(A) = \frac{\text{No. of aces of diamonds}}{\text{No. in pack}} = \frac{1}{52}$$

$$P(B) = \frac{\text{No. of spades}}{\text{No. in pack}} = \frac{13}{52}$$

$$P(A \text{ and } B) = \frac{\text{No. of spades which are also aces of diamonds}}{\text{No. in pack}} = \frac{1}{52}$$

$$\therefore P(\text{spade or ace of diamonds}) = \frac{1}{52} + \frac{13}{52} = \frac{14}{52} = \frac{7}{26}$$

(b) At a local shop 50% of customers buy unwrapped bread and 60% buy wrapped bread. What proportion of customers buy at least one kind of bread if 20% buy both wrapped and unwrapped bread?

Let S represent all the customers.

Let T represent those customers buying unwrapped bread.

Let W represent those customers buying wrapped bread.

$$P(\text{buy at least one kind of bread}) = P(\text{buy wrapped or unwrapped or both})$$

$$= P(T \text{ or } W)$$

$$= P(T) + P(W) - P(T \text{ and } W)$$

$$= 0.5 + 0.6 - 0.2$$

$$= 0.9$$

So, 9/10 of the customers buy at least one kind of bread.

Multiplication Law for Independent Events

Consider an item on a production line. This item could be defective or acceptable. These two possibilities are mutually exclusive and represent a complete list of alternatives. Assume that:

Probability that it is defective, $P(D) = 0.2$

Probability that it is acceptable, $P(A) = 0.8$.

Now consider another facet of these items. There is a system for checking them, but only every tenth item is checked. This is shown as:

Probability that it is checked $P(C) = 0.1$

Probability that it is not checked $P(N) = 0.9$.

Again these two possibilities are mutually exclusive and they represent a complete list of alternatives. An item is either checked or it is not.

Consider the possibility that an individual item is both defective and not checked. These two events can obviously both occur together so they are not mutually exclusive. They are, however, independent. That is to say, whether an item is defective or acceptable does not affect the probability of it being tested.

There are also other kinds of independent events. If you toss a coin once and then again a second time, the outcome of the second test is independent of the results of the first one. The

results of any third or subsequent test are also independent of any previous results. The probability of heads on any test is 0.5 even if all the previous tests have resulted in heads.

To work out the probability of two independent events both happening, you use the multiplication law. This can be stated as:

$$P(A \text{ and } B) = P(A) \times P(B) \text{ if } A \text{ and } B \text{ are independent events.}$$

Again this result is true for any number of independent events.

$$\text{So } P(A \text{ and } B \text{ and } \dots \text{ and } N) = P(A) \times P(B) \times \dots \times P(N).$$

Consider the example above. For any item:

$$\text{Probability that it is defective, } P(D) = 0.2$$

$$\text{Probability that it is acceptable, } P(A) = 0.8$$

$$\text{Probability that it is checked, } P(C) = 0.1$$

$$\text{Probability that it is not checked, } P(N) = 0.9.$$

Using the multiplication law to calculate the probability that an item is both defective and not checked

$$P(D \text{ and } N) = 0.2 \times 0.9 = 0.18.$$

The probabilities of the other combinations of independent events can also be calculated.

$$P(D \text{ and } C) = 0.2 \times 0.1 = 0.02$$

$$P(A \text{ and } N) = 0.8 \times 0.9 = 0.72$$

$$P(A \text{ and } C) = 0.8 \times 0.1 = 0.08.$$

Examples

- a) A machine produces two batches of items. The first batch contains 1,000 items of which 20 are damaged. The second batch contains 10,000 items of which 50 are damaged. If one item is taken from each batch, what is the probability that both items are defective?

For the item from the first batch:

$$\text{Probability that it is defective} \quad P(D_1) = \frac{20}{1,000} = 0.02$$

For the item taken from the second batch:

$$\text{Probability that it is defective} \quad P(D_2) = \frac{50}{1,000} = 0.005$$

Since these two probabilities are independent

$$P(D_1 \text{ and } D_2) = P(D_1) \times P(D_2) = 0.02 \times 0.005 = 0.0001.$$

- b) A card is drawn at random from a well shuffled pack of playing cards. What is the probability that the card is a heart? What is the probability that the card is a three? What is the probability that the card is the three of hearts?

$$\text{Probability of a heart, } P(H) = \frac{13}{52}$$

$$\text{Probability of a three, } P(3) = \frac{4}{52}$$

Probability of the three of hearts:

$$P(H \text{ and } 3) = P(H) \times P(3) = \frac{13}{52} \times \frac{4}{52} = \frac{1}{52}$$

since the suit and the number of a card are independent.

- c) A dice is thrown three times. What is the probability of one or more sixes in these three throws?

$$\text{Probability of no six in first throw} = \frac{5}{6}$$

$$\text{Similarly, probability of no six in second or third throw} = \frac{5}{6}$$

The result of each throw is independent, so

$$\text{Probability of no six in all three throws} = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$$

Since no sixes and one or more sixes are mutually exclusive and cover all possibilities,

$$\text{Probability of one or more sixes} = 1 - \frac{125}{216} = \frac{91}{216}$$

Distinguishing the Laws

Although the above laws of probability are not complicated, you must think carefully and clearly when using them. Remember that events must be mutually exclusive before you can use the addition law, and they must be independent before you can use the multiplication law. Another matter about which you must be careful is the listing of equally likely outcomes. Be sure that you list all of them. For example, we can list the possible results of tossing two coins, namely:

First Coin	Second Coin
Heads	Heads
Tails	Heads
Heads	Tails
Tails	Tails

There are four equally likely outcomes. Do not make the mistake of saying, for example, that there are only two outcomes (both heads or not both heads); you must list all the possible outcomes. (In this case “not both heads” can result in three different ways, so the probability of this result will be higher than “both heads”.)

In this example, the probability that there will be one heads and one tails (heads - tails, or tails - heads) is 0.5. This is a case of the addition law at work, the probability of heads - tails ($1/4$) plus the probability of tails - heads ($1/4$). Putting it another way, the probability of different faces is equal to the probability of the same faces - in both cases $1/2$.

D. TREE DIAGRAMS

A compound experiment, i.e. one with more than one component part, may be regarded as a sequence of similar experiments. For example, the rolling of two dice can be considered as the rolling of one followed by the rolling of the other; and the tossing of four coins can be thought of as tossing one after the other. A tree diagram enables us to construct an exhaustive list of mutually exclusive outcomes of a compound experiment.

Furthermore, a tree diagram gives us a pictorial representation of probability.

By **exhaustive**, we mean that every possible outcome is considered.

By **mutually exclusive** we mean, as before, that if one of the outcomes of the compound experiment occurs then the others cannot.

Examples

- a) The concept can be illustrated using the example of a bag containing five red and three white billiard balls. If two are selected at random without replacement, what is the probability that one of each colour is drawn?

We can represent this as a tree diagram as in Figure 1.

N.B. R indicates red ball

W indicates white ball.

Probabilities at each stage are shown alongside the branches of the tree.

Figure 1.1

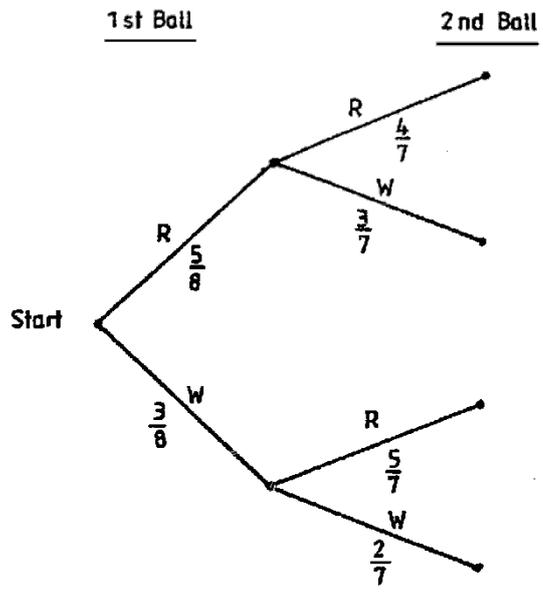


Table 1.1

Outcome	Probability		
RR	$\frac{5}{8}$	$\times \frac{4}{7}$	$= \frac{20}{56}$
RW	$\frac{5}{8}$	$\times \frac{3}{7}$	$= \frac{15}{56}$
WR	$\frac{3}{8}$	$\times \frac{5}{7}$	$= \frac{15}{56}$
WW	$\frac{3}{8}$	$\times \frac{2}{7}$	$= \frac{6}{56}$
Total	1		

We work from left to right in the tree diagram. At the start we take a ball from the bag. This ball is either red or white so we draw two branches labelled R and W, corresponding to the two possibilities. We then also write on the branch the probability of the outcome of this simple experiment being along that branch.

Thus for the first ball: $P(R) = \frac{5}{8}$ and $P(W) = \frac{3}{8}$ as there were five red and three white balls in the bag.

We then consider drawing a second ball from the bag. Whether we draw a red or a white ball the first time, we can still draw a red or a white ball the second time, so we mark in the two possibilities at the end of each of the two branches of our existing tree diagram. We can then see that there are four different mutually exclusive outcomes possible, namely RR, RW, WR and WW. We enter on these second branches the conditional probabilities associated with them.

Thus, on the uppermost branch in the diagram we must insert the probability of obtaining a second red ball given that the first was red. This probability is $\frac{4}{7}$ as there are only seven balls left in the bag, of which four are red. Similarly for the other branches.

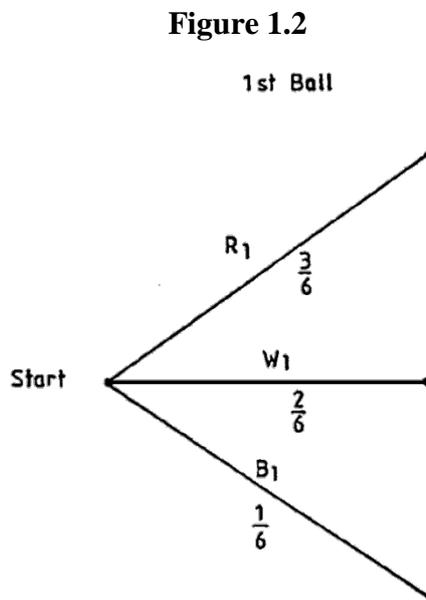
Each complete branch from start to tip represents one possible outcome of the compound experiment and each of the branches is mutually exclusive. To obtain the probability of a particular outcome of the compound experiment occurring, we multiply the probabilities along the different sections of the branch, using the general multiplication law for probabilities.

We thus obtain the probabilities shown in Table 1.1. The sum of the probabilities should add up to 1, as we know one or other of these mutually exclusive outcomes is certain to happen.

b) A bag contains three red balls, two white balls and one blue ball. Two balls are drawn at random (without replacement). Find the probability that:

- i. Both white balls are drawn.
- ii. The blue ball is not drawn.
- iii. A red then a white are drawn.
- iv. A red and a white are drawn.

To solve this problem, let us build up a tree diagram.



The first ball drawn has a subscript of 1, e.g. red first = R_1 . The second ball drawn has a subscript of 2.

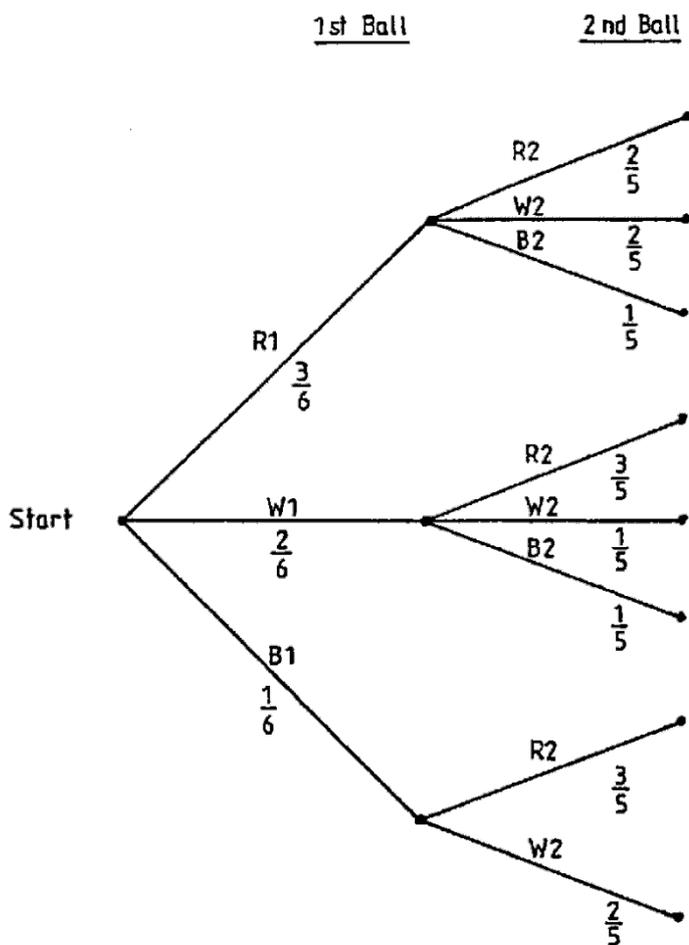
$$P(R_1) = \frac{\text{No. of red balls}}{\text{Total no. of balls}} = \frac{3}{6}$$

$$P(W_1) = \frac{2}{6}$$

$$P(B_1) = \frac{1}{6}$$

Note there is only one blue ball in the bag, so if we picked a blue ball first then we can have only a red or a white second ball. Also, whatever colour is chosen first, there are only five balls left as we do not have replacement

Figure 1.3



We can now list all the possible outcomes, with their associated probabilities:

Table 1.2

Outcome	Probability
R_1R_2	$\frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$
R_1W_2	$\frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$
R_1B_2	$\frac{3}{6} \times \frac{1}{5} = \frac{1}{10}$
W_1R_2	$\frac{2}{6} \times \frac{3}{5} = \frac{1}{5}$
W_1W_2	$\frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$
W_1B_2	$\frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$
B_1R_2	$\frac{1}{6} \times \frac{3}{5} = \frac{1}{10}$
B_1W_2	$\frac{1}{6} \times \frac{2}{5} = \frac{1}{15}$
Total	= 1

It is possible to read off the probabilities we require from Table 1.2.

(i) Probability that both white balls are drawn:

$$= P(W_1W_2) = \frac{1}{15}$$

(ii) Probability the blue ball is not drawn:

$$\begin{aligned} &= P(R_1R_2) + P(R_1W_2) + P(W_1R_2) + P(W_1W_2) \\ &= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{15} = \frac{2}{3} \end{aligned}$$

Probability that a red **then** a white are drawn:

$$= P(R_1W_2) = \frac{1}{5}$$

Probability that a red **and** a white are drawn:

$$\begin{aligned} &= P(R_1W_2) + P(W_1R_2) \\ &= \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \end{aligned}$$

- c) A couple go on having children, to a maximum of four, until they have a son. Draw a tree diagram to find the possible families' size and calculate the probability that they have a son.

We assume that any one child is equally likely to be a boy or a girl, i.e. $P(B) = P(G) = 1/2$. Note that once they have produced a son, they do not have any more children. The tree diagram will be as in Figure 1.4.

Figure 1.4

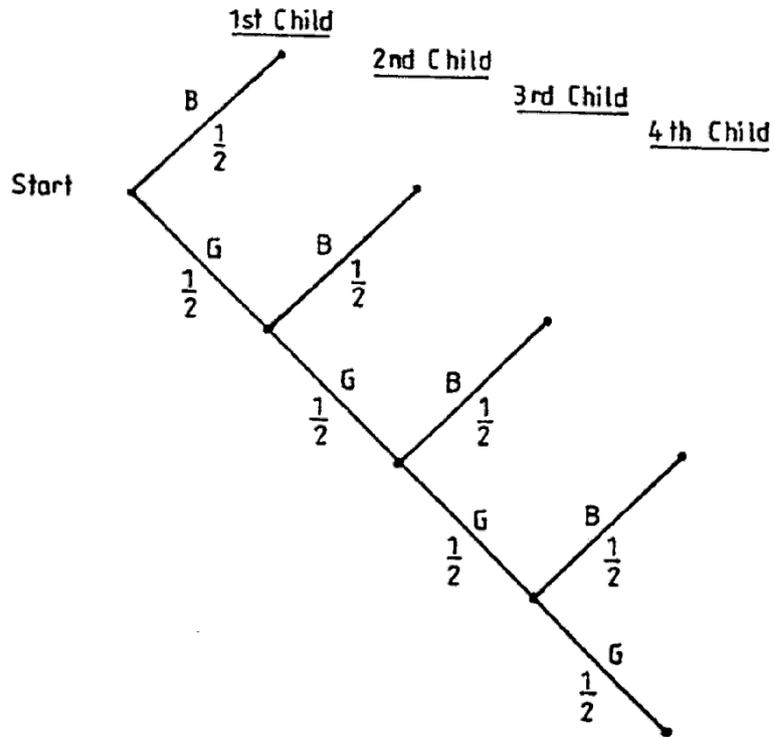


Table 1.3

Possible Families	Probability
1 Boy	$\frac{1}{2}$
1 Girl, 1 Boy	$(\frac{1}{2})^2 = \frac{1}{4}$
2 Girls, 1 Boy	$(\frac{1}{2})^3 = \frac{1}{8}$
3 Girls, 1 Boy	$(\frac{1}{2})^4 = \frac{1}{16}$
4 Girls	$(\frac{1}{2})^4 = \frac{1}{16}$
Total	= 1

Probability they have a son is therefore:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

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E. BINOMIAL DISTRIBUTION

The binomial distribution can be used to describe the likely outcome of events for discrete variables which:

- (a) Have only two possible outcomes; and
- (b) Are independent.

Suppose we are conducting a questionnaire. The *Binomial distribution* might be used to analyse the results if the only two responses to a question are ‘yes’ or ‘no’ and if the response to one question (eg, ‘yes’) does not influence the likely response to any other question (ie ‘yes’ and ‘no’).

Put rather more formally, the *Binomial distribution* occurs when there are n independent trials (or tests) with the probability of ‘success’ or ‘failure’ in each trial (or test) being constant.

Let p = the probability of ‘success’

Let q = the probability of ‘failure’

Then $q = 1 - p$

For example, if we toss an unbiased coin ten times, we might wish to find the probability of getting four heads! Here $n = 10$, p (head) = 0.5, q (tail) = 0.5 and $q = 1 - p$.

The probability of obtaining r ‘successes’ in ‘ n ’ trials (tests) is given by the following formula:

$${}_n C_r = \frac{n!}{(n-r)!r!}$$

where C is the number of combinations.

The probability of getting exactly four heads out of ten tosses of an unbiased coin, can therefore be solved as:

$$P(4) = {}_{10}C_4 0.5^4 0.5^6$$

$$\text{now } {}_{10}C_4 = \frac{10!}{(10-4)!4!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

$$\text{so } P(4) = 210 \times (0.5)^4 \times (0.5)^6$$

$$P(4) = 210 \times 0.625 \times 0.015625$$

$$P(4) = 0.2051$$

In other words the probability of getting exactly four heads out of ten tosses of an unbiased coin is 0.2051 or 20.51%.

It may be useful to state the formulae for finding *all* the possible probabilities of obtaining r successes in n trials.

Where $P(r) = {}_n C_r p^r q^{n-r}$

And $r = 0, 1, 2, 3, \dots, n$

then, from our knowledge of combinations

$$P(0) = q^n$$

$$P(1) = npq^{n-1}$$

$$P(2) = \frac{n(n-1)}{2 \times 1} p^2 q^{n-2}$$

$$P(3) = \frac{n(n-1)(n-2)}{3 \times 2 \times 1} p^3 q^{n-3}$$

$$P(4) = \frac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1} p^4 q^{n-4}$$

$$P(n-2) = \frac{n(n-1)}{2 \times 1} p^{n-2} q^2$$

$$P(n-1) = np^{n-1}q$$

$$P(n) = p^n$$

F. POISSON DISTRIBUTION

Introduction

The **Poisson distribution** may be regarded as a special case of the binomial distribution. As with the Binomial distribution, the Poisson distribution can be used where there are only two possible outcomes:-

1. Success (p)
2. Failure (q)

These events are independent. The Poisson distribution is usually used where n is very large but p is very small, and where the mean np is constant and typically < 5 . As p is very small ($p < 0.1$ and often much less), then the chance of the event occurring is extremely low. The Poisson distribution is therefore typically used for unlikely events such as accidents, strikes etc.

The Poisson distribution is also used to solve problems where events tend to occur at random, such as incoming phone calls, passenger arrivals at a terminal etc.

Whereas the formula for solving Binomial problems uses the probabilities, for both “success” (p) and “failure” (q), the formula for solving Poisson problems only uses the probabilities for “success” (p).

If μ is the mean, it is possible to show that the probability of r successes is given by the formula:

$$P(r) = \frac{e^{-\mu} \mu^r}{r!}$$

where e = exponential constant = 2.7183

μ = mean number of successes = np

n = number of trials

p = probability of “success”

r = number of successes

If we substitute $r = 0, 1, 2, 3, 4, 5, \dots$ in this formula we obtain the following expressions:

$$P(0) = e^{-\mu}$$

$$P(1) = \mu e^{-\mu}$$

$$P(2) = \frac{\mu^2 e^{-\mu}}{2 \times 1}$$

$$P(3) = \frac{\mu^3 e^{-\mu}}{3 \times 2 \times 1}$$

$$3 \times 2 \times 1$$

$$P(4) = \frac{\mu^4 e^{-\mu}}{4 \times 3 \times 2 \times 1}$$

$$P(5) = \frac{\mu^5 e^{-\mu}}{5 \times 4 \times 3 \times 2 \times 1}$$

In questions you are either given the mean μ or you have to find μ from the information given, which is usually data for n and p ; μ is then obtained from the relationship $\mu = np$.

You have to be able to work out e raised to a negative power.

e^{-3} is the same as $\frac{1}{e^3}$ so you can simply work this out using $\frac{1}{2.7183^3}$

Alternatively, many calculators have a key marked e^x . The easiest way to find e^{-3} on your calculator is to enter 3, press +/- key, press e key, and you should obtain 0.049787. If your calculator does not have an e key but has an x^y key, enter 2.7183, press x^y key, enter 3, press +/- key, then press = key; you should obtain 0.049786.

STUDY UNIT 2

Introduction to Statistics; Collection of Data

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A. INTRODUCTION TO STATISTICS

What do we mean by “Statistics”?

“Statistics” is a word which is used in a variety of ways and with a variety of meanings, but, in whatever way it is used, it is always concerned with numerical information. There are two particular meanings of the word which concern us, namely:

- a) The numerical facts themselves: for example, we talk of the “statistics” of steel production.
- b) The methods of analysing the facts: in this sense, “Statistics” is the title of a subject like “Arithmetic” or “Chemistry” or “Physics”; sometimes the subject is called “Statistical Method”.

As a subject, Statistics is a branch of science, a branch of science which deals with facts and figures; if you have a lot of numerical information about any topic, then statistical methods help you to extract the most value from it. Like mathematics, statistics is quite general; it does not matter what the figures are about, the methods still apply. Whether you are a businessman, a scientist or an accountant, the methods of analysing your facts and figures are very similar.

One of the main features of Statistical Methods is that it deals with things in groups rather than with individuals. In comparing, say, the height of Frenchmen with the height of Englishmen, we are concerned with Frenchmen in general and Englishmen in general, but not with Marcel and John as individuals. An insurance company, to give another example, is interested in the proportion of men (or women) who die at certain ages, but it is not concerned with the age at which John Kimuda (or Mary Kimenyi), as individuals, will die.

Importance of Statistics in Business

Many people think of Statistics as part of Economics but, as we have already mentioned, the subject is much more general than that. It is true, however, that economic and business situations very often provide the kind of data which is best analysed by statistical methods and which, without such methods, is either meaningless or misleading. For this reason it is important that anyone engaged in business or industry should have some sound knowledge of Statistics. In this way he or she will be able to use the methods of Statistics to help make decisions and also, what may sometimes be more important, he or she will know how best to make use of the services of professional statisticians.

With these considerations in mind, most professional bodies concerned with business affairs include Statistics as a subject in their examinations.

Main Stages in a Statistical Investigation

Firstly we must define the problem and decide exactly what it is we want to know or to predict. Collection of relevant data follows. The classification and analysis of this data and finally the presentation of the results completes the statistical investigation.

The Subject of Statistics

As you study the subject of Statistics, you should bear in mind the following points:

- a) Statistical methods are not a “sausage machine” giving set answers to set questions. They are more like the tools in a tool chest, and for any particular job a good deal of thought and perhaps some trial and error may be needed before the correct tool is chosen and used.
- b) In real life, statistical work often involves extensive calculations, but our purpose is to learn principles and methods rather than to do lots of arithmetic. Consequently, most

of our examples will contain relatively few figures, but remember that in practice one usually (but not always) has to apply the methods to a much larger mass of data.

- c) Some statistical methods are based on advanced mathematics, but do not be put off by that. For this course we can take the mathematics for granted or learn it as we go along, and we shall not require anything but ordinary arithmetic and some very simple algebra.

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B. COLLECTION OF DATA - PRELIMINARY CONSIDERATIONS

Even before the collection of data starts, there are some important points to consider when planning a statistical investigation. Shortly I will give you a list of these together with a few notes on each; some of them you may think obvious or trivial, but do not neglect to learn them because they are very often the points which are overlooked. Furthermore, examiners like to have lists as complete as possible when they ask for them!

What, then, are these preliminary matters?

Exact Definition of the Problem

This is necessary in order to ensure that nothing important is omitted from the enquiry, and that effort is not wasted by collecting irrelevant data. The problem as originally put to the statistician is often of a very general type and it needs to be specified precisely before work can begin.

Definition of the Units

The results must appear in comparable units for any analysis to be valid. If the analysis is going to involve comparisons, then the data must all be in the same units. It is no use just asking for “output” from several factories - some may give their answers in numbers of items, some in weight of items, some in number of inspected batches and so on.

Scope of the Enquiry

No investigation should be got under way without defining the field to be covered. Are we interested in all departments of our business, or only some? Are we to concern ourselves with our own business only, or with others of the same kind?

Accuracy of the Data

To what degree of accuracy is data to be recorded? For example, are ages of individuals to be given to the nearest year or to the nearest month or as the number of completed years? If some of the data is to come from measurements, then the accuracy of the measuring instrument will determine the accuracy of the results. The degree of precision required in an estimate might affect the amount of data we need to collect. In general, the more precisely we wish to estimate a value, the more readings we need to take.

C. TYPES OF DATA

Primary and Secondary Data

In its strictest sense, primary data is data which is both original and has been obtained in order to solve the specific problem in hand. Primary data is therefore raw data and has to be classified and processed using appropriate statistical methods in order to reach a solution to the problem.

Secondary data is any data other than primary data. Thus it includes any data which has been subject to the processes of classification or tabulation or which has resulted from the application of statistical methods to primary data, and all published statistics.

Quantitative/Qualitative Categorisation

Variables may be either quantitative or qualitative. Quantitative variables, to which we shall restrict discussion here, are those for which observations are numerical in nature. Qualitative variables have non-numeric observations, such as colour of hair, although, of course, each possible non-numeric value may be associated with a numeric frequency.

Continuous/Discrete Categorisation

Variables may be either continuous or discrete. A continuous variable may take any value between two stated limits (which may possibly be minus and plus infinity). Height, for example, is a continuous variable, because a person's height may (with appropriately accurate equipment) be measured to any minute fraction of a millimetre. A discrete variable, however, can take only certain values occurring at intervals between stated limits. For most (but not all) discrete variables, these interval values are the set of integers (whole numbers).

For example, if the variable is the number of children per family, then the only possible values are 0, 1, 2, ... etc. because it is impossible to have other than a whole number of

children. However, in Ireland, shoe sizes are stated in half-units, and so here we have an example of a discrete variable which can take the values 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, etc.

D. REQUIREMENTS OF STATISTICAL DATA

Having decided upon the preliminary matters about the investigation, the statistician must look in more detail at the actual data to be collected. The desirable qualities of statistical data are the following:

- Homogeneity
- Completeness
- Accurate definition
- Uniformity.

Homogeneity

The data must be in properly comparable units. “Five houses” means little since five dwelling houses are very different from five ancestral castles. Houses cannot be compared unless they are of a similar size or value. If the data is found not to be homogeneous, there are two methods of adjustment possible.

- a) Break down the group into smaller component groups which are homogeneous and study them separately.

- b) Standardise the data. Use units such as “output per man-hour” to compare the output of two factories of very different size. Alternatively, determine a relationship between the different units so that all may be expressed in terms of one; in food consumption surveys, for example, a child may be considered equal to half an adult.

Completeness

Great care must be taken to ensure that no important aspect is omitted from the enquiry.

Accurate Definition

Each term used in an investigation must be carefully defined; it is so easy to be slack about this and to run into trouble. For example, the term “accident” may mean quite different things to the injured party, the police and the insurance company! Watch out also, when using other people’s statistics, for changes in definition. Laws may, for example, alter the definition of an “indictable offence” or of an “unemployed person”.

Uniformity

The circumstances of the data must remain the same throughout the whole investigation. It is no use, for example, comparing the average age of workers in an industry at two different times if the age structure has changed markedly. Likewise, it is not much use comparing a firm’s profits at two different times if the working capital has changed.

E. METHODS OF COLLECTING DATA

When all the foregoing matters have been dealt with, we come to the question of how to collect the data we require. The methods usually available are as follows:

- Use of published statistics
- Personal investigation/interview
- Delegated personal investigation/interview
- Questionnaire.

Published Statistics

Sometimes we may be attempting to solve a problem that does not require us to collect new information, but only to reassemble and reanalyse data which has already been collected by someone else for some other purpose.

We can often make good use of the great amount of statistical data published by governments, the United Nations, nationalised industries, chambers of trade and commerce and so on. When using this method, it is particularly important to be clear on the definition of terms and units and on the accuracy of the data. The source must be reliable and the information up-to-date.

This type of data is sometimes referred to as secondary data in that the investigator himself has not been responsible for collecting it and it thus came to him “second-hand”. By contrast, data which has been collected by the investigator for the particular survey in hand is called primary data.

The information you require may not be found in one source but parts may appear in several different sources. Although the search through these may be time-consuming, it can lead to data being obtained relatively cheaply and this is one of the advantages of this type of data collection. Of course, the disadvantage is that you could spend a considerable amount of time looking for information which may not be available.

Another disadvantage of using data from published sources is that the definitions used for variables and units may not be the same as those you wish to use. It is sometimes difficult to establish the definitions from published information, but, before using the data, you must establish what it represent

Personal Investigation/Interview

In this method the investigator collects the data himself. The field he can cover is, naturally, limited. The method has the advantage that the data will be collected in a uniform manner and with the subsequent analysis in mind. There is sometimes a danger to be guarded against though, namely that the investigator may be tempted to select data that accords with some of his preconceived notions.

The personal investigation method is also useful if a pilot survey is carried out prior to the main survey, as personal investigation will reveal the problems that are likely to occur.

Delegated Personal Investigation/Interview

When the field to be covered is extensive, the task of collecting information may be too great for one person. Then a team of selected and trained investigators or interviewers may be used. The people employed should be properly trained and informed of the purposes of the investigation; their instructions must be very carefully prepared to ensure that the results are in accordance with the “requirements” described in the previous section of this study unit. If there are many investigators, personal biases may tend to cancel out.

Care in allocating the duties to the investigators can reduce the risks of bias. For example, if you are investigating the public attitude to a new drug in two towns, do not put investigator A to explore town X and investigator B to explore town Y, because any difference that is revealed might be due to the towns being different, or it might be due to different personal

biases on the part of the two investigators. In such a case, you would try to get both people to do part of each town.

Questionnaire

In some enquiries the data consists of information which must be supplied by a large number of people. Then a very convenient way to collect the data is to issue questionnaire forms to the people concerned and ask them to fill in the answers to a set of printed questions. This method is usually cheaper than delegated personal investigation and can cover a wider field. A carefully thought-out questionnaire is often also used in the previous methods of investigation in order to reduce the effect of personal bias.

The distribution and collection of questionnaires by post suffers from two main drawbacks:

- a) The forms are completed by people who may be unaware of some of the requirements and who may place different interpretations on the questions - even the most carefully worded ones!
- b) There may be a large number of forms not returned, and these may be mainly by people who are not interested in the subject or who are hostile to the enquiry. The result is that we end up with completed forms only from a certain kind of person and thus have a biased sample.

It is essential to include a reply-paid envelope to encourage people to respond.

If the forms are distributed and collected by interviewers, a greater response is likely and queries can be answered. This is the method used, for example, in the Population Census. Care must be taken, however, that the interviewers do not lead respondents in any way.

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F. INTERVIEWING

Advantages of Interviewing

There are many advantages of using interviewers in order to collect information.

The major one is that a large amount of data can be collected relatively quickly and cheaply. If you have selected the respondents properly and trained the interviewers thoroughly, then there should be few problems with the collection of the data.

This method has the added advantage of being very versatile since a good interviewer can adapt the interview to the needs of the respondent. Similarly, if the answers given to the questions are not clear, then the interviewer can ask the respondent to elaborate on them. When this is necessary, the interviewer must be very careful not to lead the respondent into altering rather than clarifying the original answers. The technique for dealing with this problem must be tackled at the training stage.

This “face-to-face” technique will usually produce a high response rate. The response rate is determined by the proportion of interviews that are successful.

Another advantage of this method of collecting data is that with a well-designed questionnaire it is possible to ask a large number of short questions of the respondent in one interview. This naturally means that the cost per question is lower than in any other method.

Disadvantages of Interviewing

Probably the biggest disadvantage of this method of collecting data is that the use of a large number of interviewers leads to a loss of direct control by the planners of the survey. Mistakes in selecting interviewers and any inadequacy of the training programme may not be recognised until the interpretative stage of the survey is reached. This highlights the need to train interviewers correctly. It is particularly important to ensure that all interviewers ask questions in a similar manner. Even with the best will in the world, it is possible that an inexperienced interviewer, just by changing the tone of his or her voice, may give a different emphasis to a question than was originally intended.

In spite of these difficulties, this method of data collection is widely used as questions can be answered cheaply and quickly and, given the correct approach, the technique can achieve high response rates.

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G. DESIGNING THE QUESTIONNAIRE

Principles

A "questionnaire" can be defined as "a formulated series of questions, an interrogatory" and this is precisely what it is. For a statistical enquiry, the questionnaire consists of a sheet (or possibly sheets) of paper on which there is a list of questions the answers to which will form the data to be analysed. When we talk about the "questionnaire method" of collecting data, we usually have in mind that the questionnaires are sent out by post or are delivered at people's homes or offices and left for them to complete. In fact, however, the method is very often used as a tool in the personal investigation methods already described.

The principles to be observed when designing a questionnaire are as follows:

- a) Keep it as short as possible, consistent with getting the right results.
- b) Explain the purpose of the investigation so as to encourage people to give the answers.
- c) Individual questions should be as short and simple as possible.
- d) If possible, only short and definite answers like "Yes", "No", or a number of some sort should be called for.
- e) Questions should be capable of only one interpretation.
- f) There should be a clear logic in the order in which the questions are asked.
- g) There should be no leading questions which suggest the preferred answer.
- h) The layout should allow easy transfer for computer input.
- i) Where possible, use the "alternative answer" system in which the respondent has to choose between several specified answers.
- j) The respondent should be assured that the answers will be treated confidentially and that the truth will not be used to his or her detriment.
- k) No calculations should be required of the respondent.

The above principles should always be applied when designing a questionnaire and, in addition, you should understand them well enough to be able to remember them all if you are asked for them in an examination question. They are principles and not rigid rules - often one has to go against some of them in order to get the right information. Governments can often ignore these principles because they can make the completion of the questionnaire compulsory by law, but other investigators must follow the rules as far as practicable in order

to make the questionnaire as easy to complete as possible - otherwise they will receive no replies.

An Example

An actual example of a self-completion questionnaire (Figure 7) is now shown as used by an educational establishment in a research survey. Note that, as the questionnaire is incorporated in this booklet, it does not give a true format. In practice, the questionnaire was not spread over so many pages.

QUESTIONNAIRE HEADING

<p style="text-align: center;">Please Read the Following INSTRUCTIONS Very Carefully</p> <p>(1) All of the following questions are about physical recreation activities away from home. We are interested in all physical activities, such as:</p> <p>Walking (for pleasure) Allotment gardening Sport and games Dancing Swimming and so on.</p> <p>If you are unsure as to whether any of your leisure activities come within the scope of our survey, please include them in your answers.</p> <p>(2) If you only take part in some of these activities at certain times of the year - in winter or in summer for example - please include these activities in your answers.</p> <p>(3) Wherever possible indicate your answer by putting a tick in the appropriate box (or more than one box if applicable).</p> <p>(4) If you are still at school please record only those activities which you take part in outside school hours.</p>

Figure 2.1

QUESTIONS

(1) During the past twelve months, have you taken part in any physical recreation activities (as defined on the cover)?

YES	
NO	

If the answer is NO please pass to Question 10.

(2) Are you still taking part in any of these activities (however infrequently)?

or If the activity is seasonal, such as football or cricket, do you intend to take part again?

YES	
NO	

If the answer is NO please pass to Question 10.

(3) If YES, which activities do you take part in? (If there are more than four such activities put down the four on which you spend the most time.)

Activity (Please specify)

- First
- Second
- Third
- Fourth

(4) For each activity listed above, please specify the place where you take part most frequently.

Activity Location

- First
- Second
- Third
- Fourth

(5) Please specify for each activity your usual method of travel to this place. (Tick more than one method, if necessary.)

Method	Activity			
	1st	2nd	3rd	4th
Motor				
Tain				
Bus				
Car				
Bicycle				
Walk				

Figure 2.2

FINAL PAGE OF QUESTIONNAIRE

(21) Please indicate by ticking the appropriate boxes below your age, sex and marital status.

	Age Group
	14-16 years
	17-19 "
	20-24 "
	25-29 "
	30-34 "
	35-39 "
	40-44 "
	45 and over

	Male
	Female

	Married
	Single
	Widowed

	YES
	NO

(22) Have you any children living at home?

(23) Are you in full-time employment or a full-time student?

	Full-time employment
	Full-time student
	Neither

If the answer is NEITHER please pass to Question 22.

Figure 2.3

FINAL PAGE OF QUESTIONNAIRE

(24) Where is your usual place of work or study?
(Please specify)

.....
.....
.....

(25) Are there any general comments you would like to make about the provision of recreational facilities in this area?

.....
.....
.....
.....

Thank you for your co-operation

Figure 2.4

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H. CHOICE OF METHOD

Choice is difficult between the various methods, as the type of information required will often determine the method of collection. If the data is easily obtained by automatic methods or can be observed by the human eye without a great deal of trouble, then the choice is easy. The problem comes when it is necessary to obtain information by questioning respondents. The best guide is to ask yourself whether the information you want requires an attitude or opinion or whether it can be acquired from short yes/no type or similar simple answers. If it is the former, then it is best to use an interviewer to get the information; if the latter type of data is required, then a postal questionnaire would be more useful.

Do not forget to check published sources first to see if the information can be found from data collected for another survey.

Another yardstick worth using is time. If the data must be collected quickly, then use an interviewer and a short simple questionnaire. However, if time is less important than cost, then use a postal questionnaire, since this method may take a long time to collect relatively limited data, but is cheap.

Sometimes a question in the examination paper is devoted to this subject. The tendency is for the question to state the type of information required and ask you to describe the appropriate method of data collection giving reasons for your choice.

More commonly, specific definitions and explanations of various terms, such as interviewer bias, are contained in multi-part questions.

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I. DATA COLLECTION IN AUDITING

The auditor relies very heavily on data collection to perform his or her job. It would clearly be extremely costly to do a complete (100%) check of all records relating to the particular financial period under review. Also, this is not really necessary, since the auditor does not wish to prove that the financial statements are exactly correct.

Years ago, judgmental (or non-statistical) sampling was used very widely in auditing. The sample size and composition was determined purely by the auditor, and a large proportion (20-25%) of the records were normally checked. For example, the auditor might do a complete check on March, August and November during the course of an audit for one particular year. However, as businesses have increased in both size and complexity, there has been an ever-increasing volume of relevant documentation, and this has led to a move towards statistical sampling.

Usefulness of Statistical Sampling

Statistical sampling is now used almost exclusively, and is superior because it allows the auditor to quantify the estimates and the risks involved in his or her checking. It is primarily useful where there is a large number of small items, e.g. physical check on stock items, payroll check, and petty cash vouchers. It is not useful where small numbers or unusual items are concerned, e.g. material items of capital expenditure, directors' expenses and remuneration, and non-recurring items.

Methods Used

When you come to Study Unit 3, which outlines the various methods of taking samples, remember that all these methods are applicable to the auditor. The size of the sample has to be chosen carefully, and other factors need to be considered here. For example:

- How good is the internal control?
- How material is the area to be tested?
- How much precision is required?
- What is the inherent risk for this area? For example, petty cash is high, whereas fixed assets are low risk areas.

There are three basic methods for taking samples.

- a) **Acceptance sampling** includes a pre-defined level of error. When the sample is selected, the results obtained are compared with this pre-defined level. The whole area is accepted or rejected on this basis.
- b) **Discovery sampling** involves looking for particular items. For example, when testing the standard of internal control, discovery sampling could be used to look for items which do not conform.
- c) **Estimation sampling** is the most widely used method. Here a sample is taken and the results are used to estimate the proportion or the amount prevalent in the whole population. This idea will be expanded in Study Unit 8.

STUDY UNIT 3

Sampling Methods; Tabulation and Grouping of Data

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A. INTRODUCTION TO SAMPLING METHODS

A large amount of statistics is concerned with the use of samples. Before we look closely at this important area, we must be certain of what we mean by certain terms.

Some Definitions

A **sample** is a collection of ONLY SOME of the items in which we are interested. If we are concerned with the price of apples in Rwanda on a certain day, we cannot find out the price in every shop in the country and then form a frequency distribution (a tabulation which shows the number of times each different value occurs) - the job would be far too vast. So what we do is to find out the prices in a few shops and hope that they are sufficiently representative of the whole country. The few figures that we do have by this procedure make up our "sample" and the purpose of the theory of sampling is to arrive at methods which will enable us to make our sample results as reliable as they need to be.

The **population** is ALL the items that we are interested in. In the case above, where we are concerned with the price of apples, the population is every shop where apples are being sold on that day. If we count or measure the whole population it is a CENSUS. The best known example of a census in Rwanda is the census of population which would be carried out by the National Institute of Statistics of Rwanda (NISR).

Sampling methods are useful because we can analyse measurements from the sample and thereby estimate some corresponding measure for the population. For example, the proportion of faulty items in a sample can be used as an estimate of the overall proportion defective in the population. A quantity calculated from a sample is usually called a statistic; corresponding quantities for the whole population are usually called parameters. Thus we may calculate a statistic and use it as an estimate of a parameter. The accuracy with which the sample statistics reflect the population parameters is a primary concern of the theory of sampling.

Why Use Samples?

There are several reasons why we may wish to use samples instead of a census.

- a) The effort of carrying out a complete investigation may be prohibitive.
- b) The cost of carrying out a complete survey may be greater than the value of the information collected, and that is not cost-effective.
- c) The items may have to be destroyed to obtain the data. Some tests used to check the items actually destroy them. If we tested all the production from a match factory, there would not be any matches left to sell.
- d) There may not be sufficient time to carry out a census.
- e) The information may not be available to enable a census to be carried out.

Sampling Frames

Before we can start to pick a sample, we must define, as precisely as possible, the population from which the sample is to be taken. If we are going to survey a sample of meat shops in Byumba, then we must define the population by making a list of ALL the meat shops in Byumba. If we mean to carry out a survey of dwelling houses in a particular region, then we need a list of all the streets, with their house numbers or names, in all the places in the region. Such lists of the population to be sampled are called sampling frames.

Great care must be exercised in drawing up a sampling frame because if the frame does not represent the population exactly, then the subsequent samples we draw from it will not be representative; if some items in the population are missed out of the sampling frame, those items have no chance of being included in a sample. In practice, it will be very difficult, if not impossible, to get an absolutely accurate sampling frame, but every effort should be made to do so, and any deficiencies which may be detected should be rectified or noted, so that account may be taken of them when interpreting the results of the survey.

The electoral register is often used as a sampling frame by market research companies. It can never be 100% accurate as the information on households when collected takes a

considerable amount of time to be registered and by that time the statistics may be misleading due to factors such as removals, deaths, demolitions and new buildings.

Having decided on the plan of the survey and drawn up the sampling frame, the next consideration is how to take the sample from the sampling frame. For the moment we will assume that the sample size is already decided. (We will consider the theory regarding this later on.) There are many methods of taking a sample, of which the following are the most important:

- Pure or simple random sampling
- Stratified sampling
- Systematic sampling
- Multi-stage sampling
- Cluster sampling

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B. RANDOM OR PROBABILITY SAMPLING

A very important notion is that of randomness in relation to sampling. The word “random” can be defined as "heedless; without aim, purpose or principle", but this is NOT what the statistician means by it. A sample is random in the statistical sense when, at the time of selecting an item for the sample, every member of the population stands a calculable chance of being included in the sample. All sampling theory is based on this notion and it is most important that you understand and remember it.

Pure or Simple Random Sampling

In this type of sampling, every item in the population has the SAME chance of being included in the sample. As an example, take the case of the national lottery. Here all the numbered “balls” are put into a big drum and whirled about until they are thought to be thoroughly mixed. We all hope and believe that each time the draw is made, all the “balls” in the drum stand the same chance of being picked out - there is no special bias in favour of (or against) odd numbers, even numbers, small numbers, large numbers or numbers of any other particular kind. The winning “balls”, therefore, constitute a simple random sample. Random samples are, in fact, not at all easy to ensure.

All the sampling theory which is dealt with in this course depends on the true randomness of the samples under discussion. This assumption underlies all the work we do in sampling, so please do not forget it if it is not mentioned in a particular case.

There are two main ways of drawing a simple random sample from a sampling frame, namely:

- a) Number all the items in the sampling frame. Take an equal number of cards, discs or balls and number them correspondingly. Now thoroughly shuffle the cards (or shake up the discs or balls) and pick out a number of cards equal to the sample size. The items in the sampling frame the numbers of which are the same as the numbers on the sample cards constitute the random sample we require. This procedure is sometimes called "**lottery sampling**".

- b) Number all the items in the sampling frame in an exactly equivalent way, i.e. if the population consists of 9,999 items, these would be numbered 0000, 0001, 0002 ... 0010, 0011 0098, 0099, 0100, 0101, 0999, 1000, etc. so that each item is labelled with four digits. Then use a **table of random numbers**, which are published in books of statistical tables, to select those items which are to make up the sample. These tables have been generated by a computer and to use them you decide on which line and column you are going to start and then you read off numbers systematically (in groups of four in this instance). These numbers might be, for example, 2403, 3234, 9183, and you would include in your sample those items in the sampling frame labelled with these digits.

Whenever possible, this method should be used because it has a high degree of guarantee against bias.

These methods can also be adapted to time sampling. For instance, in work study a machine might be checked at random times through the day to see whether or not it was in use.

This type of sampling is known as "activity sampling". Here the sampling frame is a complete list of, say, five minute intervals through the working day and a simple random sample of these intervals is taken as in (b).

Stratified Random Sampling

A pure random sample, taken as described above, will give a correct representation in a mathematical way of the population from which it is drawn. The results from such a sample can readily be used to make predictions about the parent population. The representativeness of the sample can, however, be improved by stratifying the sampling frame and taking separate, proportionally-sized random samples from each stratum. An example will make this clear.

Example

Suppose that you intend to make a survey of public opinion on some matter of social and economic importance in a particular town. The sampling frame could be the local list of

electors, and you may decide on a sample size of 500. A purely random sample of the whole town would represent each electoral ward in APPROXIMATELY the correct proportion - but only approximately. A more representative sample would be obtained if the sample represented the electoral wards in EXACTLY the correct proportions. The list of electors is classified by wards, and the totals in the seven wards of our imaginary town might be as shown in Table 3.1.

Table 3.1

Ward	No. of Electors	%
A	4,360	12.5
B	5,240	15
C	5,070	14.5
D	4,020	11.5
E	5,600	16
F	4,200	12
G	6,510	18.5
Total	35,000	100

To obtain a stratified random sample of 500, we take a pure random sample of 62 from Ward A (i.e. 12.5% of 500), a pure random sample of 75 from Ward B (15% of 500), and so on. The result is that each ward is represented in correct proportion in the sample but the sampling within each ward is still random.

The wards in this example are called the **strata** (each one is a stratum). Other examples of strata are age groups, income groups, etc. Of course, the difficulty and cost of determining the strata may, in some cases, preclude the use of this method.

Systematic Random Sampling

This is the procedure by which items are chosen for a sample by taking them at regular intervals throughout the sampling frame. If the sampling frame contains 10,000 items and a sample of 100 is wanted, then we can take every hundredth item down the list. Provided that the first item is taken properly at random between item No. 1 and item No. 100, the resulting sample will be a random sample. For example, if we use a table of random numbers to pick a number which lies between 01 and 99, we might get 56. For our “systematic” sample we then take, from the sampling frame, items 56, 156, 256, 356, 456 and so on until we have the required sample of one hundred. This speeds up considerably the process of choosing a sample from a large population as compared with a pure random sampling method.

There is one special danger to guard against though; we must be absolutely sure that there is no regularity in the sampling frame at intervals of 100, otherwise we get a biased sample. Yule and Kendall, in “An Introduction to the Theory of Statistics”, quote the example of a sampling frame which consists of a list of all the houses in a long street; a systematic sample of every tenth house is to be taken. If the street is divided into blocks by cross-streets at every tenth house, then there is a risk that every house in the sample may be a corner house (or a mid-block house, or some other regularly occurring type) and this would constitute a biased sample. Unless there is some special reason for using systematic sampling, it is not to be preferred to pure random sampling or stratified sampling.

Multi-Stage Sampling

This is a technique by which the advantages of random sampling are largely maintained, but the extent of the survey area is reduced. It is usually employed in order to cut down the cost of using large numbers of interviewers and of travelling over the survey area.

If a survey is to be carried out in, say, Kigali Province, then interviewers would have to travel all over the region to contact people in a random sample. To overcome this, the province is divided into districts and these districts are used as a sampling frame. The frame may, of course, be stratified into “rural”, “urban”, “municipal borough” classes, etc. Then a sample of districts is taken from this sampling frame; only the sample districts will be surveyed. This is one stage of the sampling scheme. Next, within each chosen district, sub-units, such as streets or electoral wards, are chosen (by random or stratified sampling again); this is stage two. Next, within the chosen sub-units, the persons to be interviewed are chosen, by random or

stratified sampling. The number of stages need not, of course, be three – that is merely an example.

Cluster Sampling

Cluster sampling, which is another modification of random sampling, can be used when no sampling frame of all the individual items in a population is available, but a complete sampling frame does exist for some method of grouping individual items. A simple random sample of these groupings is taken and then every item in these selected groupings is used to form the cluster sample.

For example, suppose the Rwandan Housing Authority wished to find out opinions on housing conditions from those living in its suburbs. It would be possible to make a list of all the people in the suburbs and to take a simple random sample of them, but the work involved, both in compiling the list and visiting those selected, who would be dispersed in all the suburbs, would be time-consuming. The list of people living in these suburbs would be much shorter. Once a simple random sample of these suburb areas has been taken, the interviewers have only a limited number of suburb areas to visit and the saving in time, and hence costs, are considerable.

Clustering has its disadvantages as the sample might not be representative of the population as a whole, since it is limited to certain groups. It is thus preferable to have a large number of small clusters rather than a small number of large clusters at the penultimate sampling stage.

Sequential Sampling

Sequential sampling is mainly used in testing manufactured lots or batches of goods. Frequently, a sample from a batch is to be tested to check that it conforms to a standard, and the whole batch accepted or rejected on the basis of this test. Often, the items in the sample must be tested one at a time using a lengthy time-consuming test. Before all the items have been tested, it may be possible to say whether it will pass or fail.

If at a given point it is possible to say that the batch has definitely failed, or definitely passed, testing is stopped at this point. Otherwise, the sampling and testing process continues. Therefore, in sequential sampling, the actual size of the sample is not fixed in advance, although the maximum possible sample size is fixed.

C. NON-PROBABILITY SAMPLING

Quota sampling is a sampling method which does not use a sampling frame, and such a method is referred to as a non-probability sampling method.

In surveys involving personal interviews, the technique of “quota sampling” has been developed to economise on interview costs. Briefly, the method is that the interviewer is instructed to conduct interviews until a certain number have been completed (say 250). The choice of persons to interview is left entirely to the interviewer, and he is not given a list of specific persons, sampled from the sampling frame. The respondents, however, must usually satisfy specific criteria regarding age, occupation, etc.

The risk of bias in quota sampling is quite serious, because the interviewer may subconsciously select only those people who look “friendly”; or the quota may be completed at a time and place where certain types of people predominate, e.g. women at shopping times, workers outside a factory as the shift comes out. Because the technique usually requires less interviewer effort (in travelling time, or calling back where no response is given), it is often used by agencies conducting public opinion polls.

To overcome the disadvantage of bias just mentioned, the interviewers are usually given good training and advice as to how such bias may be guarded against. Furthermore, they are normally given quotas within predetermined strata, e.g. interview 50 women at their homes, 50 women entering shops, 25 women passers-by, 100 men in the street, 25 men in shops. Nevertheless, the drawbacks are serious and quota sampling should not be used without a very close examination and appreciation of its shortcomings.

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D. OBJECTIVES OF SAMPLING

Introduction

There are various reasons why we use sampling techniques. In general terms there are three areas where samples are employed to obtain information about the population in which we are interested. The areas are:

- a) Estimating or forecasting, e.g. marketing of a product.
- b) Discovery or investigating, e.g. the effect of a high protein diet on farm animals.
- c) Accepting or quality control, e.g. accepting a batch of machined parts.

Estimating

In the business world, with which we are mainly concerned in this course, this is probably the most common reason for the use of sampling techniques. The uses are too numerous to mention them all, but a few examples are market research, analysis of sales or customers and forecasting production times or quantities.

The statistical values in which we may be interested are mean values, the variance or standard deviation, comparison of one population with another, etc. (We will come back to this subject later on.)

Discovery

There is often the need to discover what is, or is not, occurring. The accountant may wish to detect any fraud, while the production manager may want to find out if the increase in the number of machine breakdowns is significant or due to random chance. The medical researcher may be interested to discover if there is any relationship between smoking and lung cancer.

In many cases the population is too large or complex to carry out a census, so samples are collected and analysed.

Quality Control

The use of sampling in quality control is well established. The quality being surveyed may be the engineering tolerances on a precision product, the accuracy of invoices, or the lighting ability of a match.

The general idea is that a batch (or day's production, etc.), which is the population, is accepted or rejected on the information provided by a sample.

E. INTRODUCTION TO CLASSIFICATION AND TABULATION OF DATA

Having completed the survey and collected the data, we need to organise it so that we can extract useful information and then present our results. The information will very often consist of a mass of figures in no very special order. For example, we may have a card index of the 3,000 workers in a large factory; the cards are probably kept in alphabetical order of names, but they will contain a large amount of other data such as wage rates, age, sex, type of work, technical qualifications and so on. If we are required to present to the factory management a statement about the age structure of the labour force (both male and female), then the alphabetical arrangement does not help us, and no one could possibly gain any idea about the topic from merely looking through the cards as they are. What is needed is to classify the cards according to the age and sex of the worker and then present the results of the classification as a tabulation. The data in its original form, before classification, is usually known as “raw data”.

Example

We cannot, of course, give here an example involving 3,000 cards, but you ought now to follow this “shortened version” involving only a small number of items.

a) Raw Data

15 cards in alphabetical order:

Ayim, L. Mr	39	years
Balewa, W. Mrs	20	“
Buhari, A. Mr	22	“
Boro, W. Miss	22	“
Chahine, S. Miss	32	“
Diop, T. Mr	30	“
Diya, C. Mrs	37	“
Eze, D. Mr	33	“

Egwu, R. Mr	45	“
Gowon, J. Mrs	42	“
Gaxa, F. Miss	24	“
Gueye, W. Mr	27	“
Jalloh, J. Miss	28	“
Jaja, J. Mr	44	“
Jang, L. Mr	39	“

b) Classification

(i) According to Sex

Ayim, L. Mr	39 years	Balewa, W. Mrs	20 years
Buhari, A. Mr	22 “	Boro, W. Miss	22 “
Diop, T. Mr	30 “	Chahine, S. Miss	32 “
Eze, D. Mr	33 “	Diya. C. Mrs	37 “
Egwu, R. Mr	45 “	Gowon, J. Mrs	42 “
Gueye, W. Mr	27 “	Gaxa, F. Miss	24 “
Jaja, J. Mr	44 “	Jalloh, J. Miss	28 “
Jang, L. Mr	39 “		

(ii) According to Age (in Groups)

Balewa, W. Mrs	20 years	Ayim, L. Mr	39 years
Buhari, A. Mr	22 “	Chahine, S. Miss	32 “
Boro, W. Miss	22 “	Diop, T. Mr	30 “
Gaxa, F. Miss	24 “	Diya, C. Mrs	37 “
Gueye, W. Mr	27 “	Eze, D. Mr	33 “
Jalloh, J. Miss	28 “	Jang, L. Mr	39 “
Egwu, R. Mr		45 years	
Gowon, J. Mrs		42 “	
Jaja, J. Mr		44 “	

c) Tabulation

The number of cards in each group, after classification, is counted and the results presented in a table.

Table 3.2

Age Group	Sex		Total
	Male	Female	
20-29	2	4	6
30-39	4	2	6
40-49	2	1	3
Total	8	7	15

You should look through this example again to make quite sure that you understand what has been done.

You are now in a position to appreciate the purpose behind classification and tabulation - it is to condense an unwieldy mass of raw data to manageable proportions and then to present the results in a readily understandable form. Be sure that you appreciate this point, because examination questions **involving tabulation** often begin with a first part which asks, "What is the object of the tabulation of statistical data?", or words to that effect.

F. FORMS OF TABULATION

We classify the process of tabulation into Simple Tabulation and Complex or Matrix Tabulation.

Simple Tabulation

This covers only one aspect of the set of figures. The idea is best conveyed by an example. Consider the card index mentioned earlier; each card may carry the name of the workshop in which the person works. A question as to how the labour force is distributed can be answered by sorting the cards and preparing a simple table thus:

Table 3.3

Workshop	Number Employed
A	600
B	360
C	660
D	840
E	540
Total	3,000

Another question might have been, "What is the wage distribution in the works?", and the answer can be given in another simple table (see Table 3.4).

Table 3.4

Wages Group	Number of Employees
RWF140 but less than RWF160	105
RWF160 “ “ “ RWF180	510
RWF180 “ “ “ RWF200	920
RWF200 “ “ “ RWF220	1,015
RWF220 “ “ “ RWF240	300
RWF240 “ “ “ RWF260	150
Total	3,000

Note that such simple tables do not tell us very much - although it may be enough for the question of the moment.

Complex Tabulation

This deals with two or more aspects of a problem at the same time. In the problem just studied, it is very likely that the two questions would be asked at the same time, and we could present the answers in a complex table or matrix.

Table 3.5

Workshop	Wage Group (RWF per week)						Total No. Employed
	*140 - 159.99	160 - 179.99	180 - 199.99	200 - 219.99	220 - 239.99	240 - 259.99	
A	20	101	202	219	29	29	600
B	11	52	90	120	29	58	360
C	19	103	210	200	88	40	660
D	34	167	303	317	18	1	840
E	21	87	115	159	136	22	540
Total	105	510	920	1,015	300	150	3,000

Note *140 - 159.99 is the same as "140 but less than 160" and similarly for the other columns.

This table is much more informative than are the two simple tables, but it is more complicated. We could have divided the groups further into, say, male and female workers, or into age groups. In a later part of this study unit I will give you a list of the rules you should try to follow in compiling statistical tables, and at the end of that list you will find a table relating to our 3,000 workers, which you should study as you read the rules.

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G. SECONDARY STATISTICAL TABULATION

So far, our tables have merely classified the already available figures, the primary statistics, but we can go further than this and do some simple calculations to produce other figures, secondary statistics. As an example, take the first simple table illustrated above, and calculate how many employees there are on average per workshop. This is obtained by dividing the total (3,000) by the number of shops (5), and the table appears thus:

Table 3.6

Workshop	Number Employed
A	600
B	360
C	660
D	840
E	540
Total	3,000
Average number of employees per workshop	600

This average is a "secondary statistic". For another example, we may take the second simple table given above and calculate the proportion of workers in each wage group, thus:

Table 3.7

Wages Group (RWF)	Number of Employees	Proportion of Employees
RWF140 but less than RWF160	105	0.035
RWF160 " " " RWF180	510	0.170
RWF180 " " " RWF200	920	0.307
RWF200 " " " RWF220	1,015	0.338
RWF220 " " " RWF240	300	0.100
RWF240 " " " RWF260	150	0.050
Total	3,000	1.000

These proportions are "secondary statistics". In commercial and business statistics, it is more usual to use percentages than proportions; in the above tables these would be 3.5%, 17%, 30.7%, 33.8%, 10% and 5%.

Secondary statistics are not, of course, confined to simple tables, they are used in complex tables too, as in this example:

Table 3.8: Inspection Results for a Factory Product in Two Successive Years

Machine No.	Year 1			Year 2		
	Output	No. of Rejects	% of Rejects	Output	No. of Rejects	% of Rejects
1	800	40	5.0	1,000	100	10.0
2	600	30	5.0	500	100	20.0
3	300	12	4.0	900	45	5.0
4	500	10	2.0	400	20	5.0
Total	2,200	92	4.2	2,800	265	9.5
Average per machine	550	23	4.2	700	66.2	9.5

The percentage columns and the average line show secondary statistics. All the other figures are primary statistics.

Note carefully that **percentages cannot be added or averaged to get the percentage of a total or of an average.** You must work out such percentages on the **totals or averages themselves.**

Another danger in the use of percentages has to be watched, and that is that you must not forget the size of the original numbers. Take, for example, the case of two doctors dealing with a certain disease. One doctor has only one patient and he cures him - 100% success! The other doctor has 100 patients of whom he cures 80 - only 80% success! You can see how very unfair it would be on the hard-working second doctor to compare the percentages alone.

H. RULES FOR TABULATION

The Rules

There are no absolute rules for drawing up statistical tables, but there are a few general principles which, if borne in mind, will help you to present your data in the best possible way. Here they are:

- a) Try not to include too many features in any one table (say, not more than four or five) as otherwise it becomes rather clumsy. It is better to use two or more separate tables.
- b) Each table should have a clear and concise title to indicate its purpose.
- c) It should be very clear what units are being used in the table (tonnes, RWF, people, RWF000, etc.).
- d) Blank spaces and long numbers should be avoided, the latter by a sensible degree of approximation.
- e) Columns should be numbered to facilitate reference.
- f) Try to have some order to the table, using, for example, size, time, geographical location or alphabetical order.
- g) Figures to be compared or contrasted should be placed as close together as possible.
- h) Percentages should be placed near to the numbers on which they are based.
- i) Rule the tables neatly - scribbled tables with freehand lines nearly always result in mistakes and are difficult to follow. However, it is useful to draw a rough sketch first so that you can choose the best layout and decide on the widths of the columns.
- j) Insert totals where these are meaningful, but avoid "nonsense totals". Ask yourself what the total will tell you before you decide to include it. An example of such a "nonsense total" is given in the following table:

Table 3.9 : Election Results

Party	Year 1 Seats	Year 2 Seats	Total Seats
A	350	200	550
B	120	270	390
Total	470	470	940

The totals (470) at the foot of the two columns make sense because they tell us the total number of seats being contested, but the totals in the final column (550, 390, 940) are "nonsense totals" for they tell us nothing of value.

- k) If numbers need to be totalled, try to place them in a column rather than along a row for easier computation.
- l) If you need to emphasise particular numbers, then underlining, significant spacing or heavy type can be used. If data is lacking in a particular instance, then insert an asterisk (*) in the empty space and give the reasons for the lack of data in a footnote.
- m) Footnotes can also be used to indicate, for example, the source of secondary data, a change in the way the data has been recorded, or any special circumstances which make the data seem odd.

An Example of Tabulation

It is not always possible to obey all of these rules on any one occasion, and there may be times when you have a good reason for disregarding some of them. But only do so if the reason is really good - not just to save you the bother of thinking! Study now the layout of the following table (based on our previous example of 3,000 workpeople) and check through the list of rules to see how they have been applied.

Table 3.10: ABC & Co. Wage Structure of Labour Force Numbers of Persons in Specified Categories

Workshop	Wage Group (RWF per month)						% No. Employed	Total See Note A
	140 - 159.99	160 - 179.99	180 - 199.99	200 - 219.99	220 - 239.99	240 - 259.99		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
A	20	101	202	219	29	29	600	20.0
B	11	52	90	120	29	58	360	12.0
C	19	103	210	200	88	40	660	22.0
D	34	167	303	317	18	1	840	28.0
E	21	87	115	159	136	22	540	18.0
Total No. in Wage Group	105	510	920	1,015	300	150	3,000	100.0
% See Note (b)	3.5	17.0	30.7	33.8	10.0	5.0	100.0	-

Note (a) Total no. employed in workshop as a percentage of the total workforce.

Note (b) Total no. in wage group as a percentage of the total workforce.

Table 3.10 can be called a "twofold" table as the workforce is broken down by wage and workshop.

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J. SOURCES OF DATA AND PRESENTATION METHODS

Sources, nature, application and use:

Sources

Data is generally found through research or as the result of a survey. Data which is found from a survey is called primary data; it is data which is collected for a particular reason or research project. For example, if your firm wished to establish how much money tourists spend on cultural events when they come to Rwanda or how long a particular process takes on average to complete in a factory. In this case the data will be taken in raw form, i.e. lots of figures and then analysed by grouping the data into more manageable groups. The other source of data is secondary data. This is data which is already available (government statistics, company reports etc). As a business person you can take these figures and use them for whatever purpose you require.

Nature of data.

Data is classified according to the type of data it is. The classifications are as follows:

Categorical data: example: Do you currently own any stocks or bonds? Yes No

This type of data is generally plotted using a bar chart or pie chart.

Numerical data: This is usually divided into **discrete** or **continuous** data.

How many cars do you own? This is **discrete** data. This is data that arises from a counting process.

How tall are you? This is **continuous** data. This is data that arises from a measuring process. Or the figures cannot be measured precisely. For example: clock in times of the workers in a particular shift: 8:23; 8:14; 8:16....

Whether data is discrete or continuous will determine the most appropriate method of presentation.

Precaution in use.

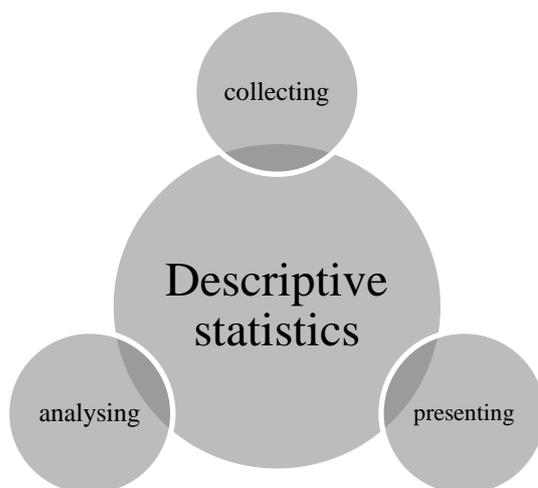
As a business person it is important that you are cautious when reading data and statistics. In order to draw intelligent and logical conclusions from data you need to understand the various meanings of statistical terms.

Role of statistics in business analysis and decision making.

In the business world, statistics has four important applications:

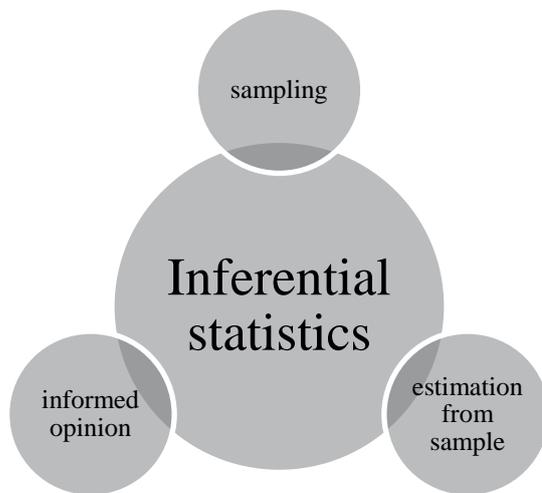
- To summarise business data
- To draw conclusions from that data
- To make reliable forecasts about business activities
- To improve business processes.

The field of statistics is generally divided into two areas.



Descriptive statistics allows you to create different tables and charts to summarise data. It also provides statistical measures such as the mean, median, mode, standard deviation etc to describe different characteristics of the data

Figure 3.1



Drawing conclusions about your data is the fundamental point of inferential statistics. Using these methods allows the researcher to draw conclusions based on data rather than on intuition.

Figure 3.2

Improving business processes involves using managerial approaches that focus on quality improvements such as Six Sigma. These approaches are data driven and use statistical method to develop these models.

- Presentation of data, use of bar charts, histograms, pie charts, graphs, tables, frequency distributions, cumulative distributions, Ogives.
- Their uses and interpretations.

If you look at any magazine or newspaper article, TV show, election campaign etc you will see many different charts depicting anything from the most popular holiday destination to the gain in company profits. The nice thing about studying statistics is that once you understand the concepts the theory remains the same for all situations and you can easily apply your knowledge to whatever situation you are in.

Tables and charts for **categorical** data:

When you have categorical data, you tally responses into categories and then present the frequency or percentage in each category in tables and charts.

The summary table indicates the frequency, amount or percentage of items in each category, so that you can differentiate between the categories.

Supposing a questionnaire asked people how they preferred to do their banking:

Table 3.11

Banking preference	frequency	percentage
In bank	200	20
ATM	250	25
Telephone	97	10
internet	450	45
Total	997	100

The above information could be illustrated using a bar chart

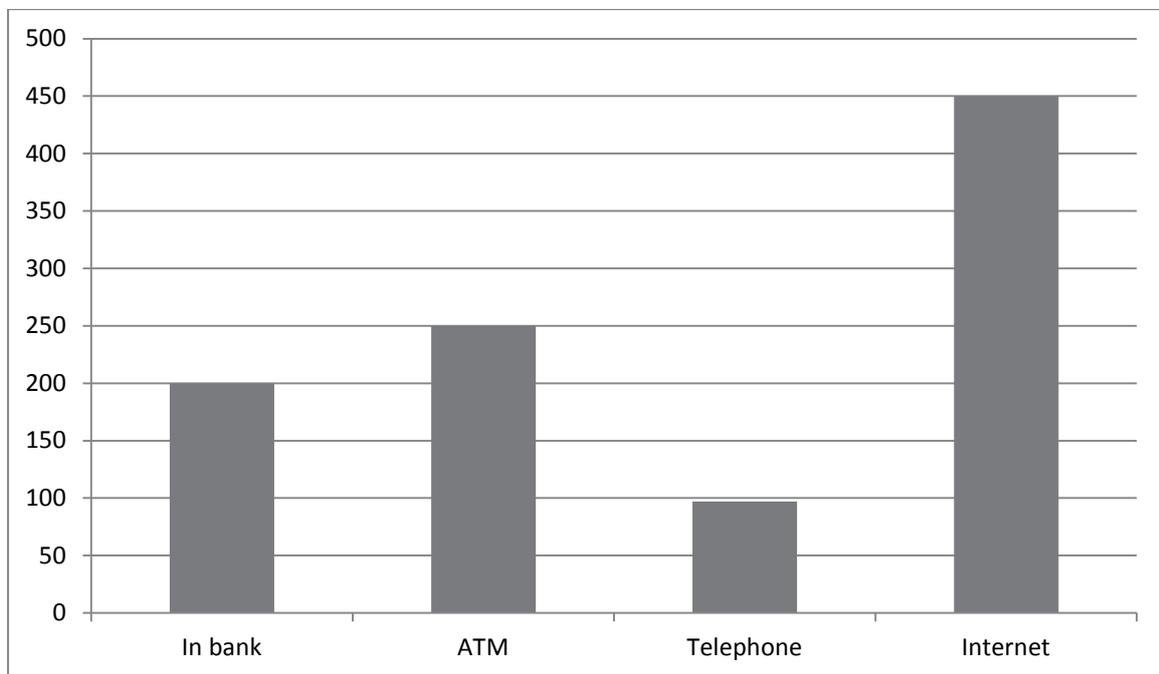


Figure 3.3

Or a pie chart

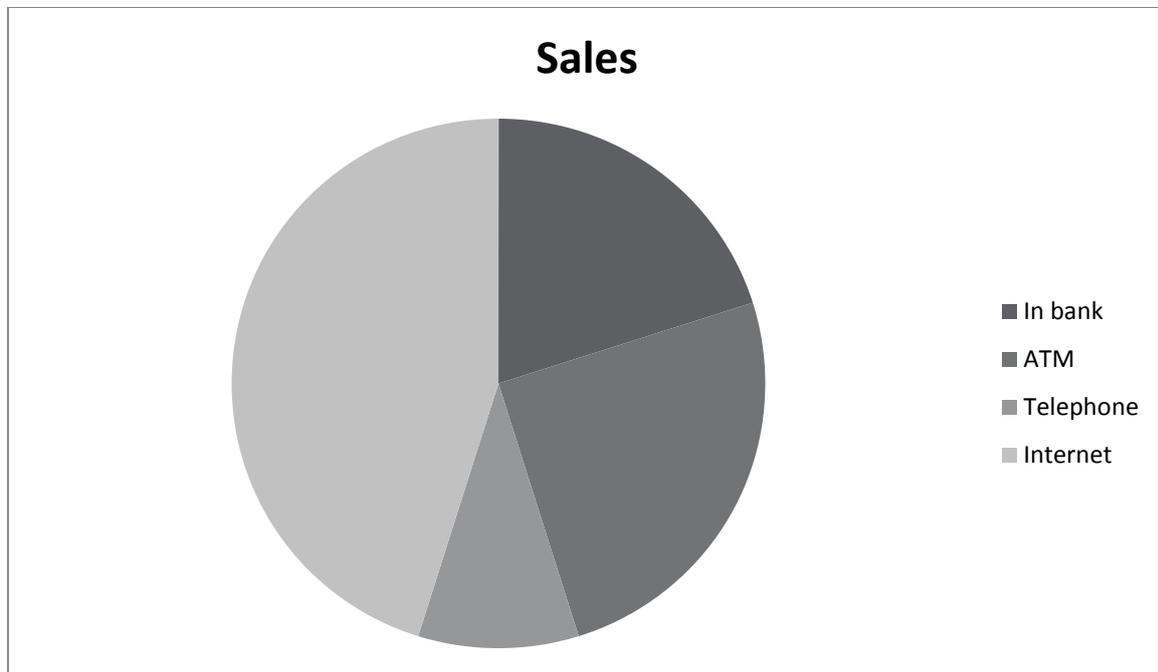


Figure 3.4

A simple line chart is usually used for time series data, where data is given over time.

The price of an average mobile homes over the past 3 years

Table 3.12

Year	Price RWF
2008	RWF350 000
2009	RWF252 000
2010	RWF190 000

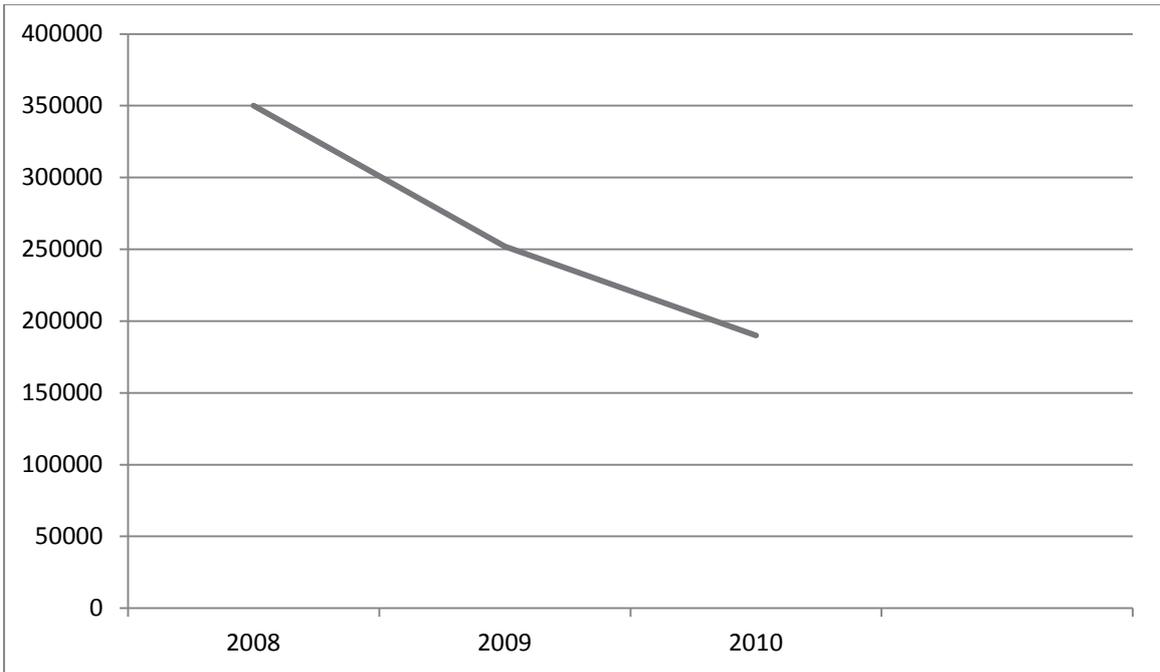


Figure 3.5

The above graphs are used for categorical data.

Numerical Data

Numerical data is generally used more in statistics. The process in which numerical data is processed is as follows.



Figure 3.6

The Histogram:

The histogram is like a bar chart but for numerical data. The important thing to remember about the histogram is that the area under the histogram represents or is proportionate to the frequencies. If you are drawing a histogram for data where the class widths are all the same then it is very easy. If however one class width is bigger or narrower than the others an adjustment must be made to ensure that the area of the bar is proportionate to the frequency.

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STUDY UNIT 4

Graphical Representation of Information

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A. INTRODUCTION TO FREQUENCY DISTRIBUTIONS

A frequency distribution is a tabulation which shows the number of times (i.e. the frequency) each different value occurs. Refer back to Study Unit 2 and make sure you understand the difference between "attributes" (or qualitative variables) and "variables" (or quantitative variables); the term "frequency distribution" is usually confined to the case of variables.

Example

The following figures are the times (in minutes) taken by a shop-floor worker to perform a given repetitive task on 20 specified occasions during the working day:

3.5	3.8	3.8	3.4	3.6
3.6	3.8	3.9	3.7	3.5
3.4	3.7	3.6	3.8	3.6
3.7	3.7	3.7	3.5	3.9

If we now assemble and tabulate these figures, we obtain a frequency distribution (see Table 4.1).

Table 4.1

Length of Time (Minutes)	Frequency
3.4	2
3.5	3
3.6	4
3.7	5
3.8	4
3.9	2
Total	20

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B. PREPARATION OF FREQUENCY DISTRIBUTIONS

Simple Frequency Distribution

A useful way of preparing a frequency distribution from raw data is to go through the records as they stand and mark off the items by the "tally mark" or "five-bar gate" method. First look at the figures to see the highest and lowest values so as to decide the range to be covered and then prepare a blank table.

Now mark the items on your table by means of a tally mark. To illustrate the procedure, the following table shows the state of the work after all 20 items have been entered.

Table 4.2

Length of Time (Minutes)	Tally Marks	Frequency
3.4		2
3.5		3
3.6		4
3.7	/	5
3.8		4
3.9		2
Total		20

Grouped Frequency Distribution

Sometimes the data is so extensive that a simple frequency distribution is too cumbersome and, perhaps, uninformative. Then we make use of a "grouped frequency distribution".

In this case, the "length of time" column consists not of separate values but of groups of values (see Table 4.3).

Table 4.3

Length of Time (Minutes)	Tally Marks	Frequency
3.4 to 3.5		5
3.6 to 3.7		9
3.8 to 3.9		6
Total		20

Grouped frequency distributions are only needed when there is a large number of values and, in practice, would not have been required for the small amount of data in our example. Table 4.4 shows a grouped frequency distribution used in a more realistic situation, when an ungrouped table would not have been of much use.

Table 4.4: Age Distribution of Workers in an Office

Age Group (Years)	No. of Workers
15 but less than 20	10
20 " " " 25	17
25 " " " 30	28
30 " " " 35	42
35 " " " 40	38
40 " " " 45	30
45 " " " 50	25
50 " " " 55	20
55 " " " 60	10
Total	220

The various groups (e.g. "25 but less than 30") are called "**classes**" and the range of values covered by a class (e.g. five years in this example) is called the "**class interval**".

The number of items in each class (e.g. 28 in the 25 to 30 class) is called the "class frequency" and the total number of items (in this example, 220) is called the "total frequency". As stated before, frequency distributions are usually only considered in

connection with variables and not with attributes, and you will sometimes come across the term "variate" used to mean the variable in a frequency distribution. The variate in our last example is "age of worker", and in the previous example the variate was "length of time".

The term "class boundary" is used to denote the dividing line between adjacent classes, so in the age group example the class boundaries are 15, 20, 25, years. In the length of time example, as grouped earlier in this section, the class boundaries are 3.35, 3.55, 3.75, 3.95 minutes. This needs some explanation. As the original readings were given correct to one decimal place, we assume that is the precision to which they were measured. If we had had a more precise stopwatch, the times could have been measured more precisely. In the first group of 3.4 to 3.5 are put times which could in fact be anywhere between 3.35 and 3.55 if we had been able to measure them more precisely. A time such as 3.57 minutes would not have been in this group as it equals 3.6 minutes when corrected to one decimal place and it goes in the 3.6 to 3.7 group.

Another term, "class limits", is used to stand for the lowest and highest values that can actually occur in a class. In the age group example, these would be 15 years and 19 years 364 days for the first class, 20 years and 24 years 364 days for the second class and so on, assuming that the ages were measured correct to the nearest day below. In the length of time example, the class limits are 3.4 and 3.5 minutes for the first class and 3.6 and 3.7 minutes for the second class.

You should make yourself quite familiar with these terms, and with others which we will encounter later, because **they are all used freely by examiners** and you will not be able to answer questions if you don't know what the questioner means!

Choice of Class Interval

When compiling a frequency distribution you should, if possible, make the length of the class interval equal for all classes so that fair comparison can be made between one class and another. Sometimes, however, this rule has to be broken (official publications often lump together the last few classes into one so as to save paper and printing costs) and then, before we use the information, it is as well to make the classes comparable by calculating a column showing "frequency per interval of so much", as in this example for some wage statistics:

Table 4.5

Annual Income (RWF00)	No. of Persons	Frequency per RWF200 Interval
48.0 but less than 50.0	50,000	50,000
50.0 “ “ “ 52.0	40,000	40,000
52.0 “ “ “ 56.0	55,000	27,500
56.0 “ “ “ 60.0	23,000	11,500
60.0 “ “ “ 64.0	12,000	6,000
64.0 “ “ “ 72.0	12,000	3,000
Total	192,000	-

Notice that the intervals in the first column are:

200, 200, 400, 400, 400, 800.

These intervals let you see how the last column was compiled.

A superficial look at the original table (first two columns only) might have suggested that the most frequent incomes were at the middle of the scale, because of the appearance of the figure 55,000. But this apparent preponderance of the middle class is due solely to the change in the length of the class interval, and column three shows that, in fact, the most frequent incomes are at the bottom end of the scale, i.e. the top of the table.

You should remember that the purpose of compiling a grouped frequency distribution is to make sense of an otherwise troublesome mass of figures. It follows, therefore, that we do not want to have too many groups or we will be little better off; nor do we want too few groups or we will fail to see the significant features of the distribution. As a practical guide, you will find that somewhere between about five and 20 groups will usually be suitable.

When compiling grouped frequency distributions, we occasionally run into trouble because some of our values lie exactly on the dividing line between two classes and we wonder which class to put them into. For example, in the age distribution given earlier in Table 24, if we

have someone aged exactly 40 years, do we put him into the "35-40" group or into the "40-45" group? There are two possible solutions to this problem:

- a) Describe the classes as "x but less than y" as we have done in Table 24, and then there can be no doubt.
- b) Where an observation falls exactly on a class boundary, allocate half an item to each of the adjacent classes. This may result in some frequencies having half units, but this is not a serious drawback in practice.

The first of these two procedures is the one to be preferred.

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C. CUMULATIVE FREQUENCY DISTRIBUTIONS

Very often we are not especially interested in the separate class frequencies, but in the number of items above or below a certain value. When this is the case, we form a cumulative frequency distribution as illustrated in column three of the following table:

Table 4.6

Length of Time (Minutes)	Frequency	Cumulative Frequency
3.4	2	2
3.5	3	5
3.6	4	9
3.7	5	14
3.8	4	18
3.9	2	20

The cumulative frequency tells us the number of items equal to or less than the specified value, and it is formed by the successive addition of the separate frequencies. A cumulative frequency column may also be formed for a grouped distribution.

The above example gives us the number of items "less than" a certain amount, but we may wish to know, for example, the number of persons having more than some quantity. This can easily be done by doing the cumulative additions from the bottom of the table instead of the top, and as an exercise you should now compile the "more than" cumulative frequency column in the above example.

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D. RELATIVE FREQUENCY DISTRIBUTIONS

All the frequency distributions which we have looked at so far in this study unit have had their class frequencies expressed simply as numbers of items. However, remember that proportions or percentages are useful secondary statistics. When the frequency in each class of a frequency distribution is given as a proportion or percentage of the total frequency, the result is known as a "**relative frequency distribution**" and the separate proportions or percentages are the "relative frequencies". The total relative frequency is, of course, always 1.0 (or 100%). Cumulative relative frequency distributions may be compiled in the same way as ordinary cumulative frequency distributions.

As an example, the distribution used in Table 4.5 is now set out as a relative frequency distribution for you to study.

Table 4.7

Annual Income (RWF00)		Freq.	Rel. Freq. (%)	Cum. Freq.	Rel. Cum. Freq. (%)
48.0 but less than	50.0	50,000	26	50,000	26
50.0 " " "	52.0	40,000	21	90,000	47
52.0 " " "	56.0	55,000	29	145,000	76
56.0 " " "	60.0	23,000	12	168,000	88
60.0 " " "	64.0	12,000	6	180,000	94
64.0 " " "	72.0	12,000	6	192,000	100
Totals		192,000	100%		

This example is in the "less than" form, and you should now compile the "more than" form in the same way as you did for the non-relative distribution.

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E. GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTIONS

Tabulated frequency distributions are sometimes more readily understood if represented by a diagram. Graphs and charts are normally much superior to tables (especially lengthy complex tables) for showing general states and trends, but they cannot usually be used for accurate analysis of data. The methods of presenting frequency distributions graphically are as follows:

- Frequency dot diagram
- Frequency bar chart
- Frequency polygon
- Histogram
- Ogive.

We will now examine each of these in turn.

Frequency Dot Diagram

This is a simple form of graphical representation for the frequency distribution of a discrete variate. A horizontal scale is used for the variate and a vertical scale for the frequency. Above each value on the variate scale we mark a dot for each occasion on which that value occurs. Thus, a frequency dot diagram of the distribution of times taken to complete a given task, which we have used in this study unit, would look like Figure 4.1.

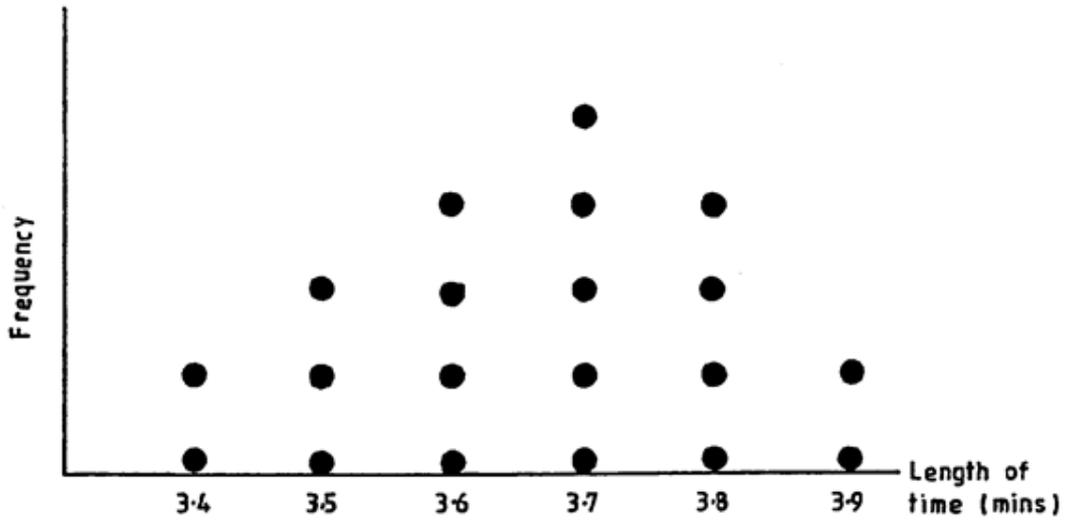


Figure 4.1: Frequency Dot Diagram to Show Length of Time Taken by Operator to Complete a Given Task

Frequency Bar Chart

We can avoid the business of marking every dot in such a diagram by drawing instead a vertical line the length of which represents the number of dots which should be there. The frequency dot diagram in Figure 4.1 now becomes a frequency bar chart, as in Figure 4.2.

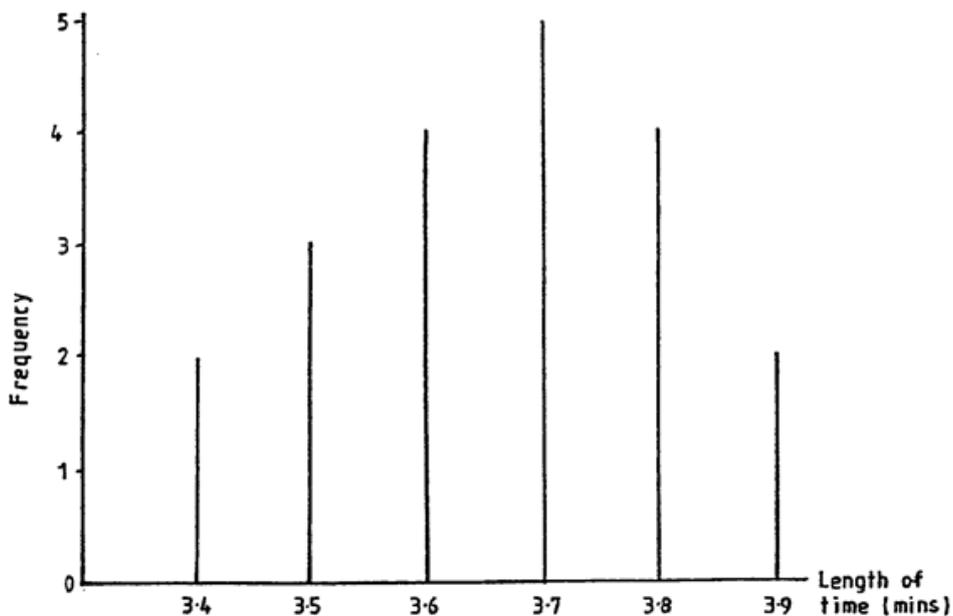


Figure 4.2: Frequency Bar Chart

Frequency Polygon

Instead of drawing vertical bars as we do for a frequency bar chart, we could merely mark the position of the top end of each bar and then join up these points with straight lines. When we do this, the result is a frequency polygon, as in Figure 4.3.

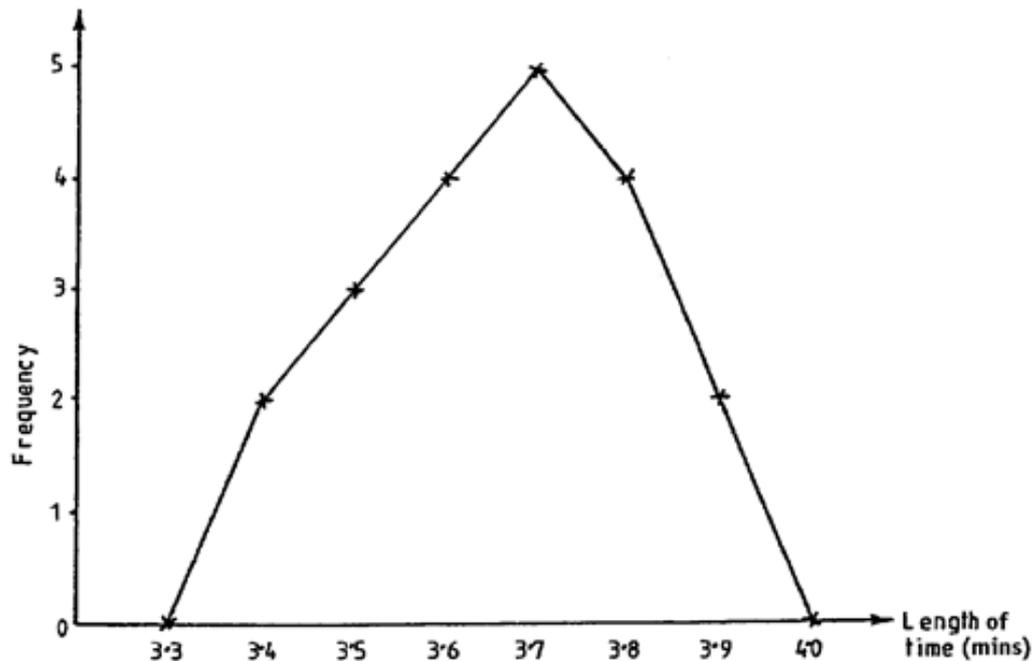


Figure 4.3: Frequency Polygon

Note that we have added two fictitious classes at each end of the distribution, i.e. we have marked in groups with zero frequency at 3.3 and 4.0.

This is done to ensure that the area enclosed by the polygon and the horizontal axis is the same as the area under the corresponding **histogram** which we shall consider in the next section.

These three kinds of diagram are all commonly used as a means of making frequency distributions more readily comprehensible. They are mostly used in those cases where the variate is discrete and where the values are **not grouped**. Sometimes frequency bar charts and polygons are used with grouped data by drawing the vertical line (or marking its top end) at the centre point of the group.

Histogram

This is the best way of graphing a grouped frequency distribution. It is of great practical importance and is also a favourite topic among examiners. Refer back now to the grouped distribution given earlier in Table 4.4 (ages of office workers) and then study Figure 4.5.

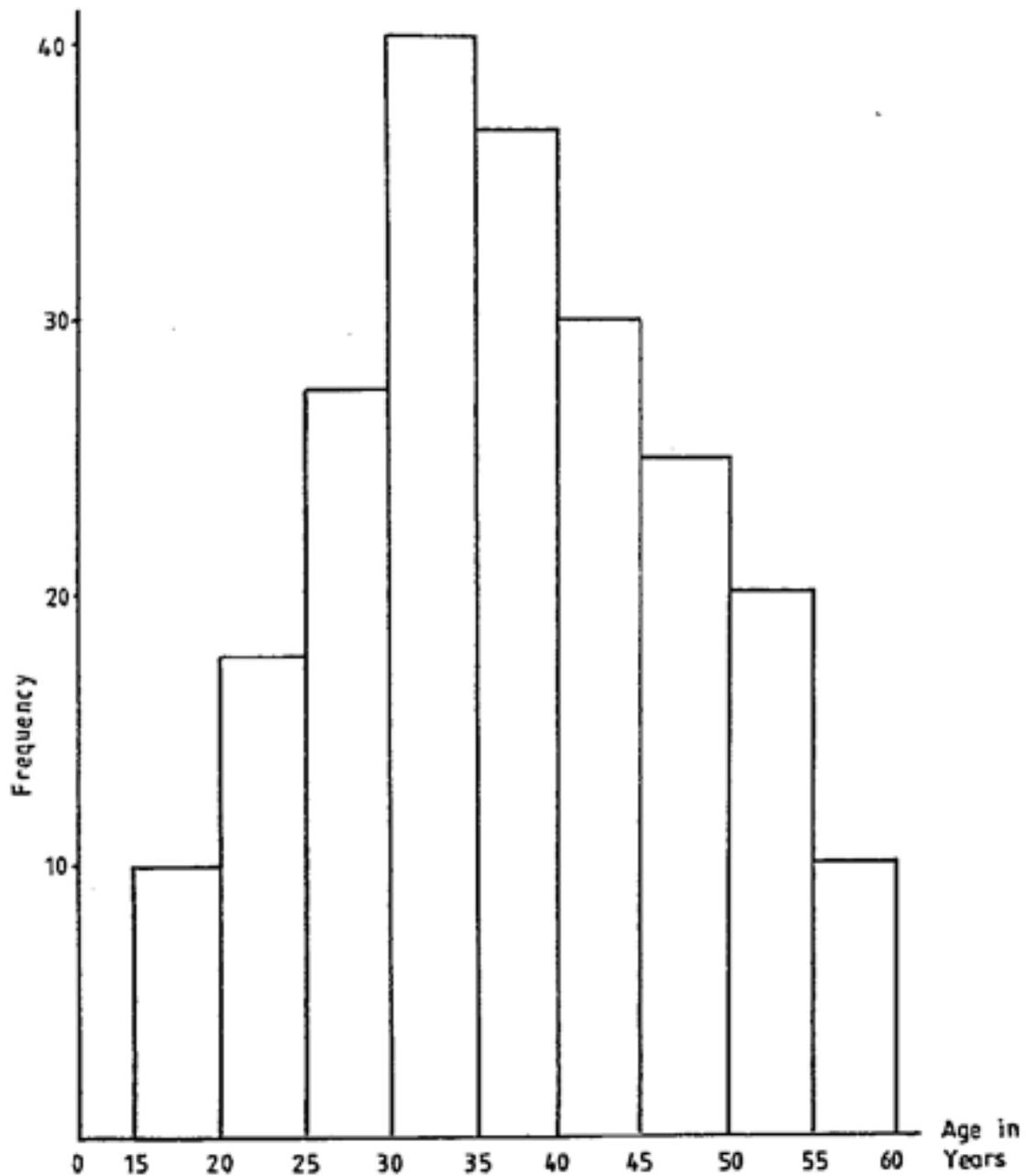


Figure 4.5: Histogram

We call this kind of diagram a "histogram". The frequency in each group is represented by a rectangle and - this is a very important point - it is the **AREA of the rectangle, not its height, which represents the frequency.**

When the lengths of the class intervals are all equal, then the heights of the rectangles represent the frequencies in the same way as do the areas (this is why the vertical scale has been marked in this diagram); if, however, the lengths of the class intervals are not all equal, you must remember that the heights of the rectangles have to be adjusted to give the correct areas. Do not stop at this point if you have not quite grasped the idea, because it will become clearer as you read on.

Look once again at the histogram of ages given in Figure 4.5 and note particularly how it illustrates the fact that the frequency falls off towards the higher age groups - any form of graph which did not reveal this fact would be misleading. Now let us imagine that the original table had NOT used equal class intervals but, for some reason or other, had given the last few groups as:

Table 4.8

Age Group (Years)	Number of People
40 but less than 45	30
45 " " " 50	25
50 " " " 60	30

The last two groups have been lumped together as one. A **WRONG** form of histogram, using heights instead of areas, would look like Figure 4.6.

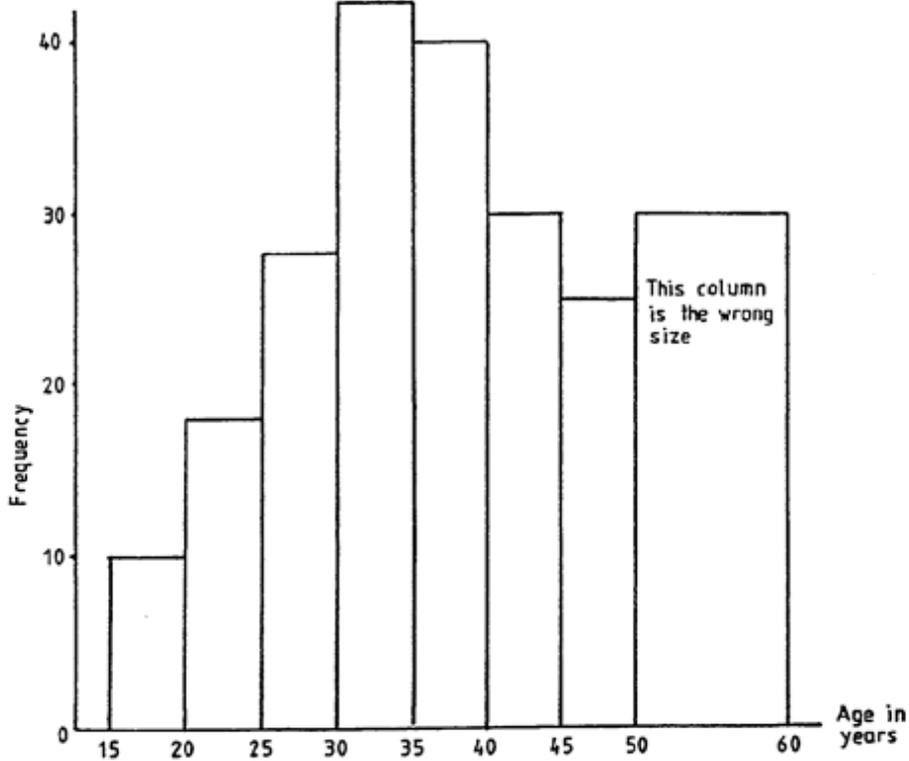


Figure 4.6

Now, this clearly gives an entirely wrong impression of the distribution with respect to the higher age groups. In the correct form of the histogram, the height of the last group (50-60) would be halved because the class interval is double all the other class intervals. The histogram in Figure 4.7 gives the right impression of the falling off of frequency in the higher age groups. I have labelled the vertical axis "Frequency density per 5-year interval" as five years is the "standard" interval on which we have based the heights of our rectangles.

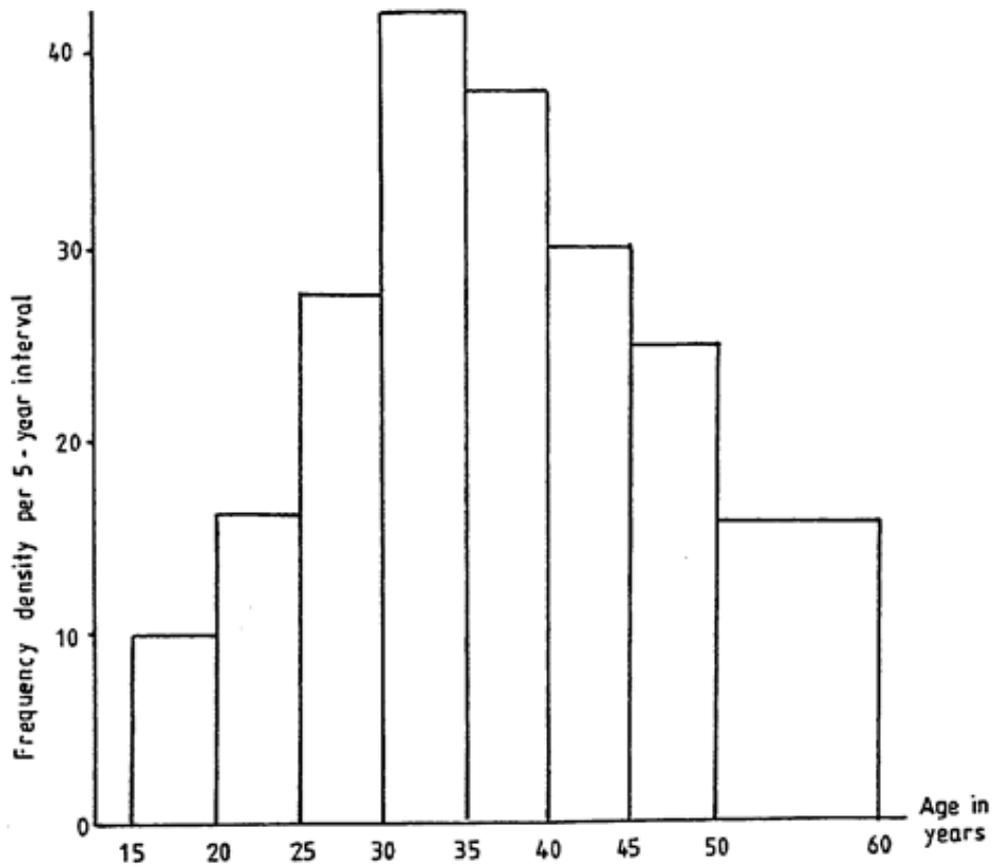


Figure 4.7

Often it happens, in published statistics, that the last group in a frequency table is not completely specified. The last few groups may look as in Table 4.9:

Table 4.9

Age Group (Years)	Number of People
40 but less than 45	30
45 " " " 50	25
50 and over 30	

How do we draw the last group on the histogram?

If the last group has a very small frequency compared with the total frequency (say, less than about 1% or 2%) then nothing much is lost by leaving it off the histogram altogether. If the last group has a larger frequency than about 1% or 2%, then you should try to judge from the general shape of the histogram how many class intervals to spread the last frequency over in order not to create a false impression of the extent of the distribution. In the example given, you would probably spread the last 30 people over two or three class intervals but it is often simpler to assume that an open-ended class has the same length as its neighbour. Whatever procedure you adopt, the important thing in an examination paper is to state clearly what you have done and why. A distribution of the kind we have just discussed is called an "**open-ended**" distribution.

The Ogive

This is the name given to the graph of the cumulative frequency. It can be drawn in either the "less than" or the "or more" form, but the "less than" form is the usual one. Ogives for two of the distributions already considered in this study unit are now given as examples; Figure 4.8 is for ungrouped data and Figure 4.9 is for grouped data.

Study these two diagrams so that you are quite sure that you know how to draw them. There is only one point which you might be tempted to overlook in the case of the grouped distribution - the points are plotted at the ends of the class intervals and NOT at the centre point. Look at the example and see how the 168,000 is plotted against the **upper end** of the 56-60 group and not against the mid-point, 58. If we had been plotting an "or more" ogive, the plotting would have to have been against the lower end of the group.

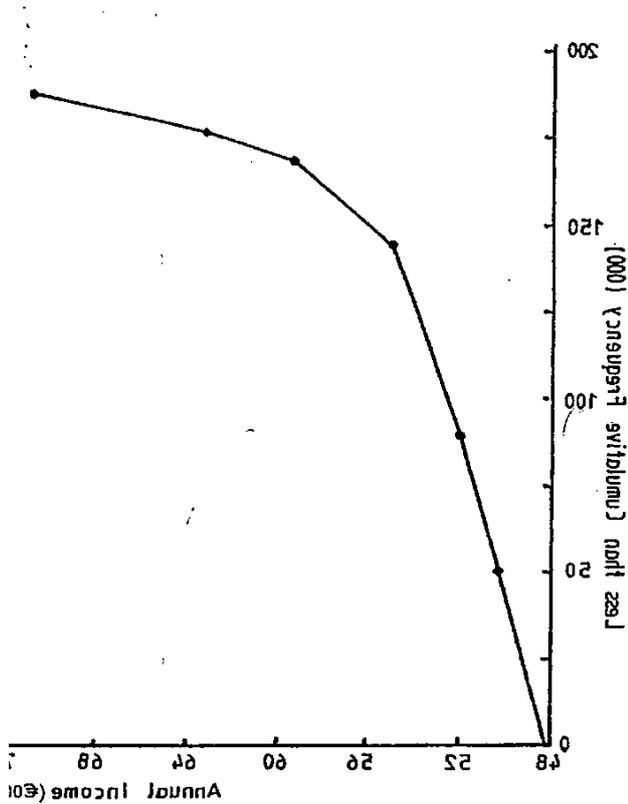


Figure 4.8

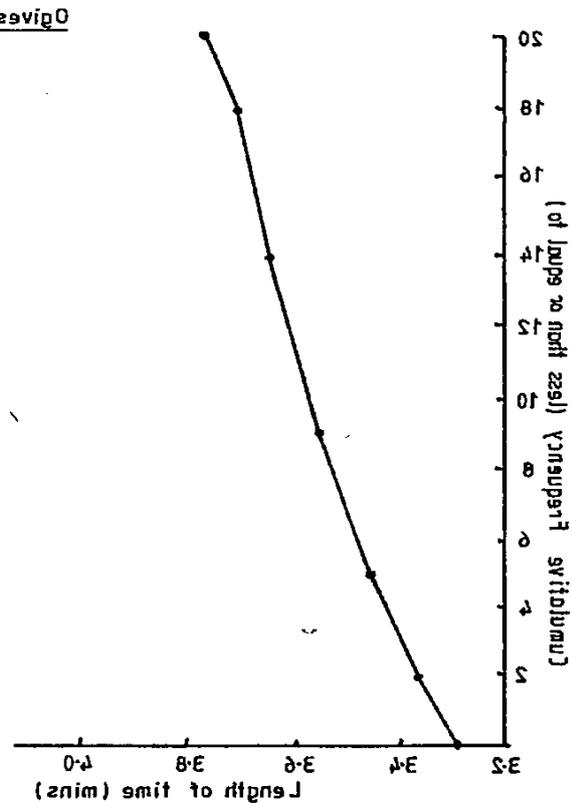


Figure 4.9

As an example of an "or more" ogive, we will compile the cumulative frequency of our example from Section B, which for convenience is repeated below with the "more than" cumulative frequency:

Table 4.10

Annual Income (RWF 000)	Number of Persons	CUMULATIVE FREQUENCY (MORE THAN)
48.0 but less than 50.0	50,000	192,000
50.0 " " " 52.0	40,000	142,000
52.0 " " " 56.0	55,000	102,000
56.0 " " " 60.0	23,000	47,000
60.0 " " " 64.0	12,000	24,000
64.0 " " " 72.0	12,000	12,000
Total	192,000	

The ogive now appears as shown in Figure 4.10

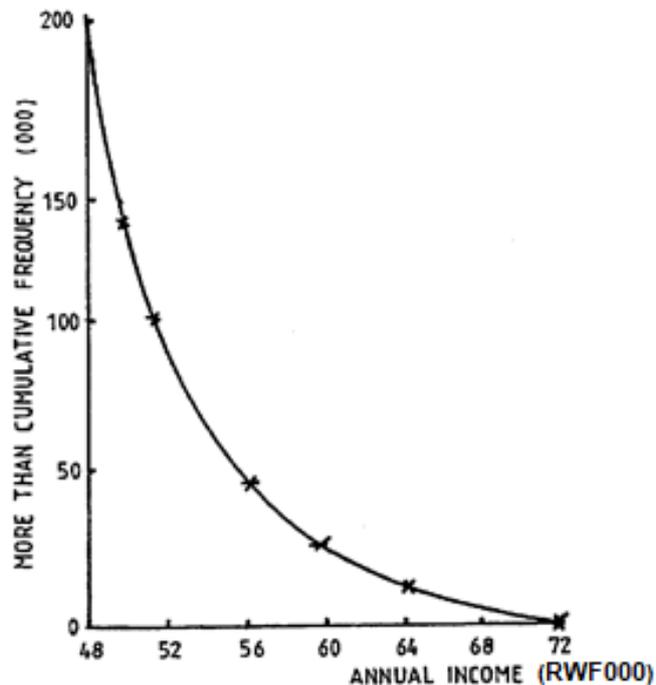


Figure 4.10

Check that you see how the plotting has been made against the lower end of the group and notice how the ogive has a reversed shape.

In each of Figures 4.9 and 4.10 we have added a fictitious group of zero frequency at one end of the distribution.

It is common practice to call the cumulative frequency graph a cumulative frequency polygon if the points are joined by straight lines, and a cumulative frequency curve if the points are joined by a smooth curve.

(N.B. Unless you are told otherwise, always compile a "less than" cumulative frequency.)

All of these diagrams, of course, may be drawn from the original figures or on the basis of relative frequencies. In more advanced statistical work the latter are used almost exclusively and you should practise using relative frequencies whenever possible.

F. INTRODUCTION TO OTHER TYPES OF DATA PRESENTATION

The graphs we have seen so far in this study unit are all based on frequency distributions. Next we shall discuss several common graphical presentations that are designed more for the lay reader than someone with statistical knowledge. You will certainly have seen some examples of them used in the mass media of newspapers and television.

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G. PICTOGRAMS

Introduction

This is the simplest method of presenting information visually. These diagrams are variously called "pictograms", "ideograms", "picturegrams" or "isotypes" - the words all refer to the same thing. Their use is confined to the simplified presentation of statistical data for the general public. Pictograms consist of simple pictures which represent quantities. There are two types and these are illustrated in the following examples. The data we will use is shown in Table 4.11.

Table 4.11: Cruises Organised by a Shipping Line Between Year 1 and Year 3

Year	Number of Cruises	No. of Passengers Carried
1	200	100,000
2	250	140,000
3	300	180,000

Limited Form

- a) We could represent the number of cruises by ships of varying size, as in Figure 4.11.

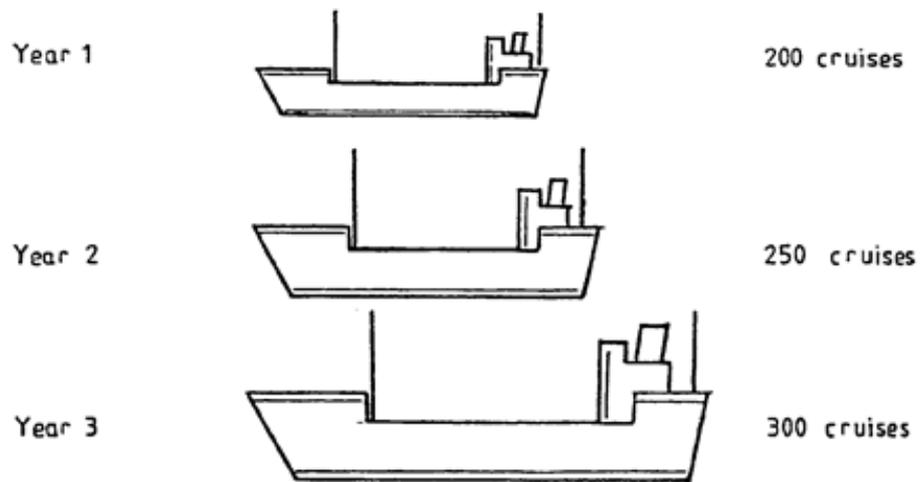


Figure 4.11: Number of Cruises Years 1-3
(Source: Table 4.11)

- b) Although these diagrams show that the number of cruises has increased each year, they can give false impressions of the actual increases. The reader can become confused as to whether the quantity is represented by the length or height of the pictograms, their area on the paper, or the volume of the object they represent. It is difficult to judge what increase has taken place. Sometimes you will find pictograms in which the sizes shown are actually **WRONG** in relation to the real increases. To avoid confusion, I recommend that you use the style of diagram shown in Figure 4.12.

Accurate Form

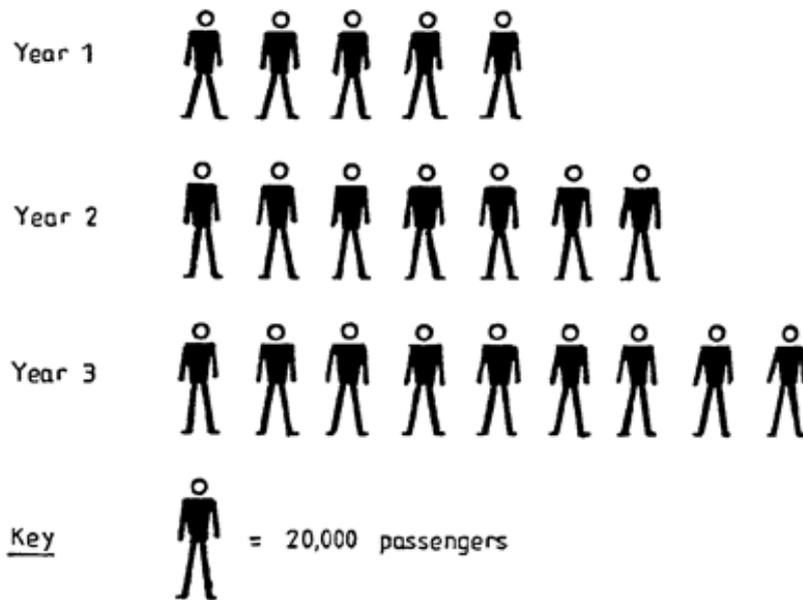


Figure 4.12: Passengers Carried Years 1-3
(Source: Table 4.11)

Each matchstick man is the same height and represents 20,000 passengers, so there can be no confusion over size.

These diagrams have no purpose other than generally presenting statistics in a simple way. Look at Figure 4.13.

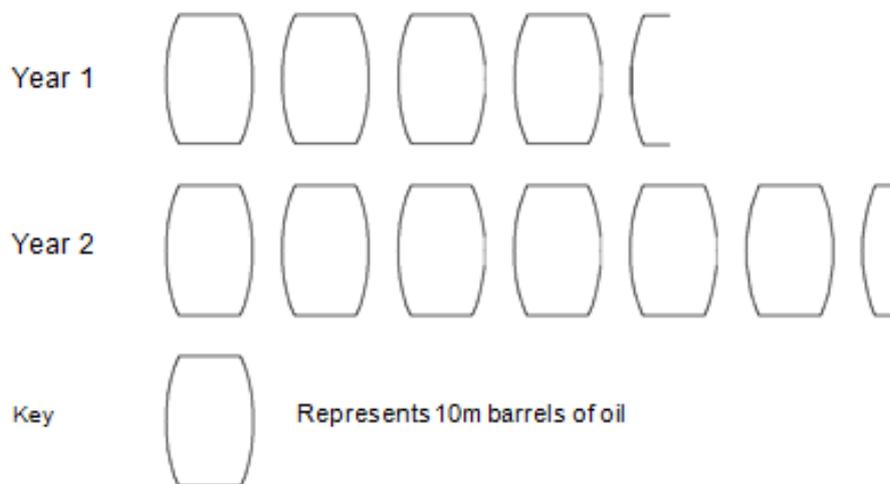


Figure 4.13: Imports of Crude Oil

Here it is difficult to represent a quantity less than 10m barrels, e.g. does "[" represent 0.2m or 0.3m barrels?

H. PIE CHARTS

These diagrams, known also as circular diagrams, are used to show the manner in which various components add up to a total. Like pictograms, they are only used to display very simple information to non-expert readers. They are popular in computer graphics.

An example will show what the pie chart is. Suppose that we wish to illustrate the sales of gas in Rwanda in a certain year. The figures are shown in Table 4.12.

Table 4.12: Gas Sales in Rwanda in One Year

Uses	Million Therms	%
Domestic	1,383	51
Industrial	843	31
Commercial	437	16
Public*	60	2
Total	2,723	100

*Central and local government uses, including public lighting

The figures are illustrated in the pie chart or circular diagram in Figure 4.14.

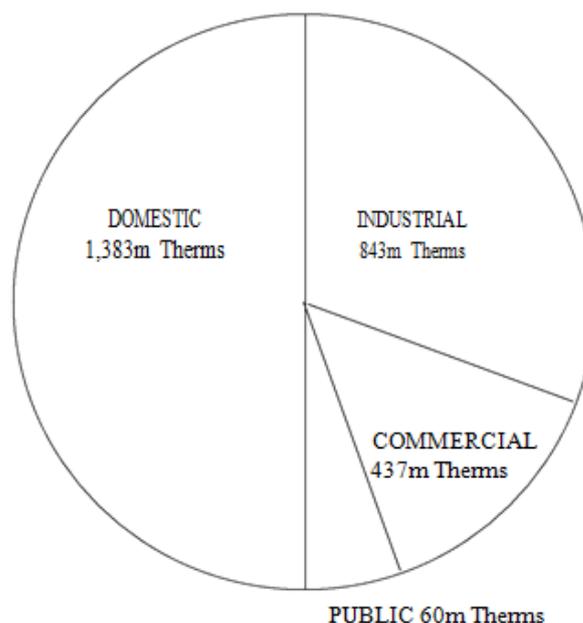


Figure 4.14: Example of a Pie Chart (Gas Sales in Rwanda)
(Source: Table 4.12)

The rules to follow are:

a) Tabulate the data and calculate the percentages.

b) Convert the percentages into degrees, e.g.

$$51\% \text{ of } 360^\circ = \frac{51}{100} = 183.6^\circ, \text{ etc.}$$

c) Construct the diagram by means of a pair of compasses and a protractor. Don't overlook this point, because examiners dislike inaccurate and roughly drawn diagrams.

d) Label the diagram clearly, using a separate "legend" or "key" if necessary. (A key is illustrated in Figure 21.)

e) If you have the choice, don't use a diagram of this kind with more than four or five component parts.

Note: The actual number of therms can be inserted on each sector as it is not possible to read this exactly from the diagram itself.

The main use of a pie chart is to show the relationship each component part bears to the whole. They are sometimes used side by side to provide comparisons, but this is not really to be recommended, unless the whole diagram in each case represents exactly the same total amount, as other diagrams (such as bar charts, which we discuss next) are much clearer. However, in examinations you may be asked specifically to prepare such pie charts.

J. BAR CHARTS

We have already met one kind of bar chart in the course of our studies of frequency distributions, namely the frequency bar chart. A "bar" is simply another name for a thick line. In a frequency bar chart the bars represent, by their length, the frequencies of different values of the variate. The idea of a bar chart can, however, be extended beyond the field of frequency distributions, and we will now illustrate a number of the types of bar chart in common use. I say "illustrate" because there are no rigid and fixed types, but only general ideas which are best studied by means of examples. You can supplement the examples in this study unit by looking at the commercial pages of newspapers and magazines.

Component Bar Chart

This first type of bar chart serves the same purpose as a circular diagram and, for that reason, is sometimes called a "component bar diagram" (see Figure 4.15).

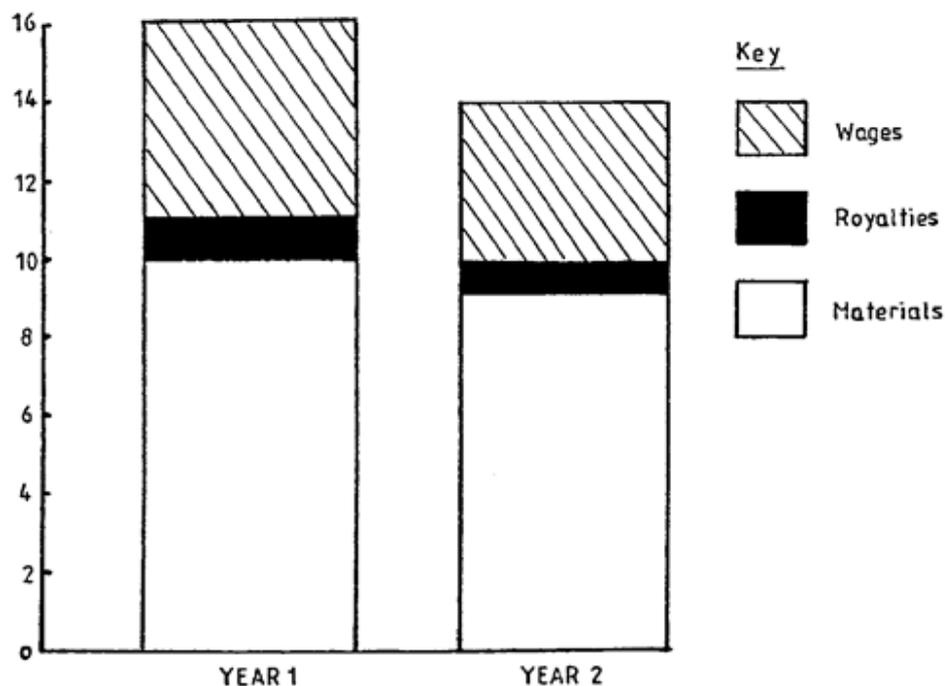


Figure 4.15: Component Bar Chart Showing Cost of Production of ZYX Co. Ltd

Note that the lengths of the components represent the amounts, and that the components are drawn in the same order so as to facilitate comparison. These bar charts are preferable to circular diagrams because:

- a) They are easily read, even when there are many components.
- b) They are more easily drawn.
- c) It is easier to compare several bars side by side than several circles.

Bar charts with vertical bars are sometimes called "column charts" to distinguish them from those in which the bars are horizontal (see Figure 4.16).

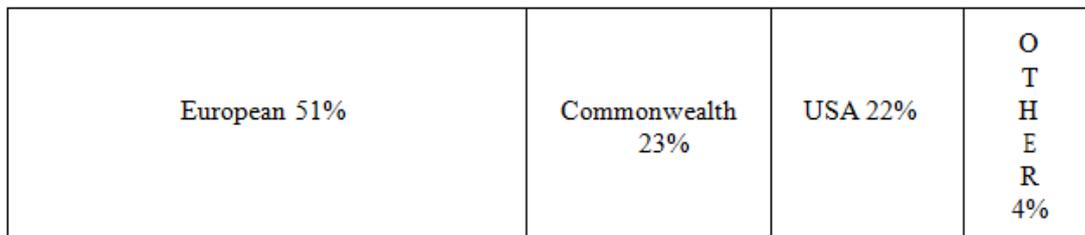


Figure 4.16: Horizontal Bar Chart of Visitors Arriving in Rwanda in One Year

Figure 4.16 is also an example of a percentage component bar chart, i.e. the information is expressed in percentages rather than in actual numbers of visitors.

If you compare several percentage component bar charts, you must be careful. Each bar chart will be the same length, as they each represent 100%, but they will not necessarily represent the same actual quantities, e.g. 50% might have been 1 million, whereas in another year it may have been nearer to 4 million and in another to 8 million.

Horizontal Bar Chart

A typical case of presentation by a horizontal bar chart is shown in Figure 4.17. Note how a loss is shown by drawing the bar on the other side of the zero line.

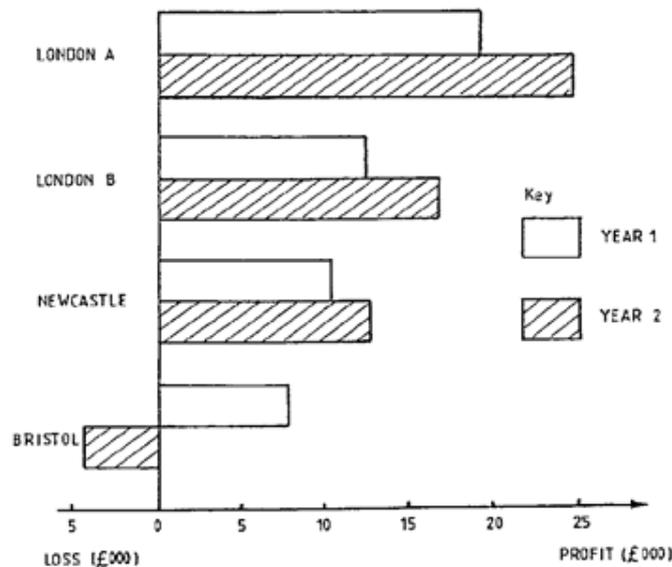


Figure 4.17: Horizontal Bar Chart for the So and So Company Ltd to Show Profits Made by Branches in Year 1 and Year 2

Pie charts and bar charts are especially useful for "categorical" variables as well as for numerical variables. The example in Figure 4.17 shows a categorical variable, i.e. the different branches form the different categories, whereas in Figure 4.15 we have a **numerical** variable, namely, time. Figure 4.17 is also an example of a multiple or compound bar chart as there is more than one bar for each category.

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K. GENERAL RULES FOR GRAPHICAL PRESENTATION

There are a number of general rules which must be borne in mind when planning and using graphical methods:

- a) Graphs and charts must be given clear but brief titles.
- b) The axes of graphs must be clearly labelled, and the scales of values clearly marked.
- c) Diagrams should be accompanied by the original data, or at least by a reference to the source of the data.
- d) Avoid excessive detail, as this defeats the object of diagrams.
- e) Wherever necessary, guidelines should be inserted to facilitate reading.
- f) Try to include the origins of scales. Obeying this rule sometimes leads to rather a waste of paper space. In such a case the graph could be "broken" as shown in Figure 4.18, but take care not to distort the graph by over-emphasising small variations.

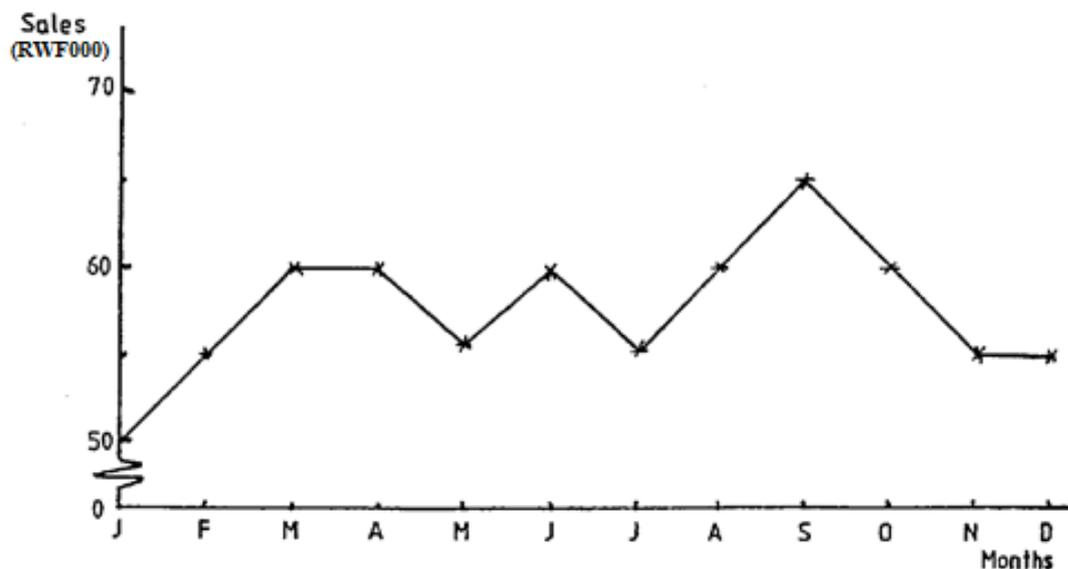


Figure 4.18

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L. THE LORENZ CURVE

Purpose

One of the problems which frequently confronts the statistician working in economics or industry is that of CONCENTRATION. Suppose that, in a business employing 100 men, the total weekly wages bill is RWF10,000 and that every one of the 100 men gets RWF100; there is then an **equal distribution** of wages and there is no concentration. Suppose now that, in another business employing 100 men and having a total weekly wages bill of RWF10,000, there are 12 highly skilled experts getting RWF320 each and 88 unskilled workers getting RWF70 each. The wages are not now equally distributed and there is some concentration of wages in the hands of the skilled experts. These experts number 12 out of 100 people (i.e. they constitute 12% of the labour force); their share of the total wages bill is $12 \times \text{RWF}320$ (i.e. RWF3,840) out of RWF10,000, which is 38.4%. We can therefore say that 38.4% of the firm's wages is concentrated in the hands of only 12% of its employees.

In the example just discussed there were only two groups, the skilled and the unskilled. In a more realistic case, however, there would be a larger number of groups of people with different wages, as in the following example:

Wages Group (RWF)	Number of People	Total Wages (RWF)
0 - 80	205	10,250
80 - 120	200	22,000
120 - 160	35	4,900
160 - 200	30	5,700
200 - 240	20	4,400
240 - 280	<u>10</u>	<u>2,500</u>
	<u>500</u>	<u>49,750</u>

Obviously when we have such a set of figures, the best way to present them is to graph them, which I have done in Figure 4.19. Such a graph is called a LORENZ CURVE. (The next section shows how we obtain this graph.)

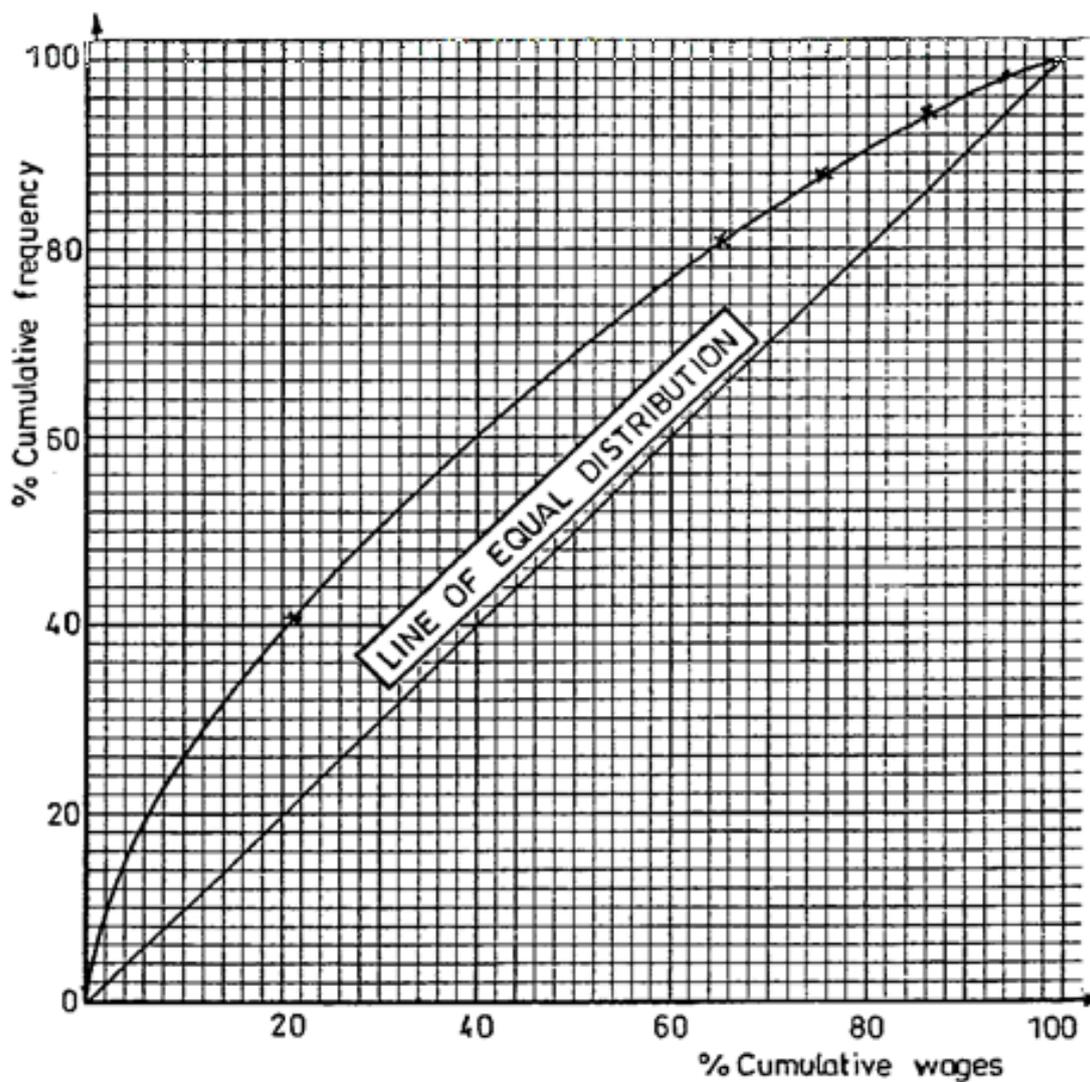


Figure 4.19: Lorenz Curve

Stages in Construction of a Lorenz Curve

- a) Draw up a table giving:
- (i) the cumulative frequency;
 - (ii) the percentage cumulative frequency;
 - (iii) the cumulative wages total;
 - (iv) the percentage cumulative wages total.

Table 4.13

Wages Group (RWF)	Number of People (Frequency)	Cumulative Frequency	% Cumulative Frequency	Total Wages (RWF)	Cumulative Wages Total (RWF)	% Cumulative Wages Total
0 - 80	205	205	41	10,250	10,250	21
80 - 120	200	405	81	22,000	32,250	65
120 - 160	35	440	88	4,900	37,150	75
160 - 200	30	470	94	5,700	42,850	86
200 - 240	20	490	98	4,400	47,250	95
240 - 280	10	500	100	2,500	49,750	100
	500			49,750		

- b) On graph paper draw scales of 0-100% on both the horizontal and vertical axes. The scales should be the same on both axes.
- c) Plot the cumulative percentage frequency against the cumulative percentage wages total and join up the points with a smooth curve. Remember that 0% of the employees earn 0% of the total wages so that the curve will always go through the origin.
- d) Draw in the 45° diagonal. Note that, if the wages had been equally distributed, i.e. 50% of the people had earned 50% of the total wages, etc., the Lorenz curve would have been this diagonal line.

The graph is shown in Figure 4.19.

Sometimes you will be given the wages bill as a grouped frequency distribution alone, without the total wages for each group being specified. Consider the following set of figures:

Wages Group (RWF)	No. of People
0 - 40	600
40 - 80	250
80 - 120	100
120 - 160	30
160 - 200	<u>20</u>
	<u>1,000</u>

As we do not know the actual wage of each person, the total amount of money involved in each group is estimated by multiplying the number of people in the group by the mid-value of the group; for example, the total amount of money in the "RWF40-RWF80" group is $250 \times \text{RWF60} = \text{RWF15,000}$. The construction of the table and the Lorenz curve then follows as before. Try working out the percentages for yourself first and then check your answers with the following table. Your graph should look like Figure 4.20.

Table 4.14

Wages Group (RWF)	Number of People	Cumulative Frequency	% Cumulative Frequency	Estimated Total Wages (RWF)	Cumulative Total Wages (RWF)	% Cumulative Total Wages
0 - 40	600	600	60	12,000	12,000	26.8
40 - 80	250	850	85	15,000	27,000	60.3
80 - 120	100	950	95	10,000	37,000	82.6
120 - 160	30	980	98	4,200	41,200	92.0
160 - 200	20	1,000	100	3,600	44,800	100.0
1,000	44,800					

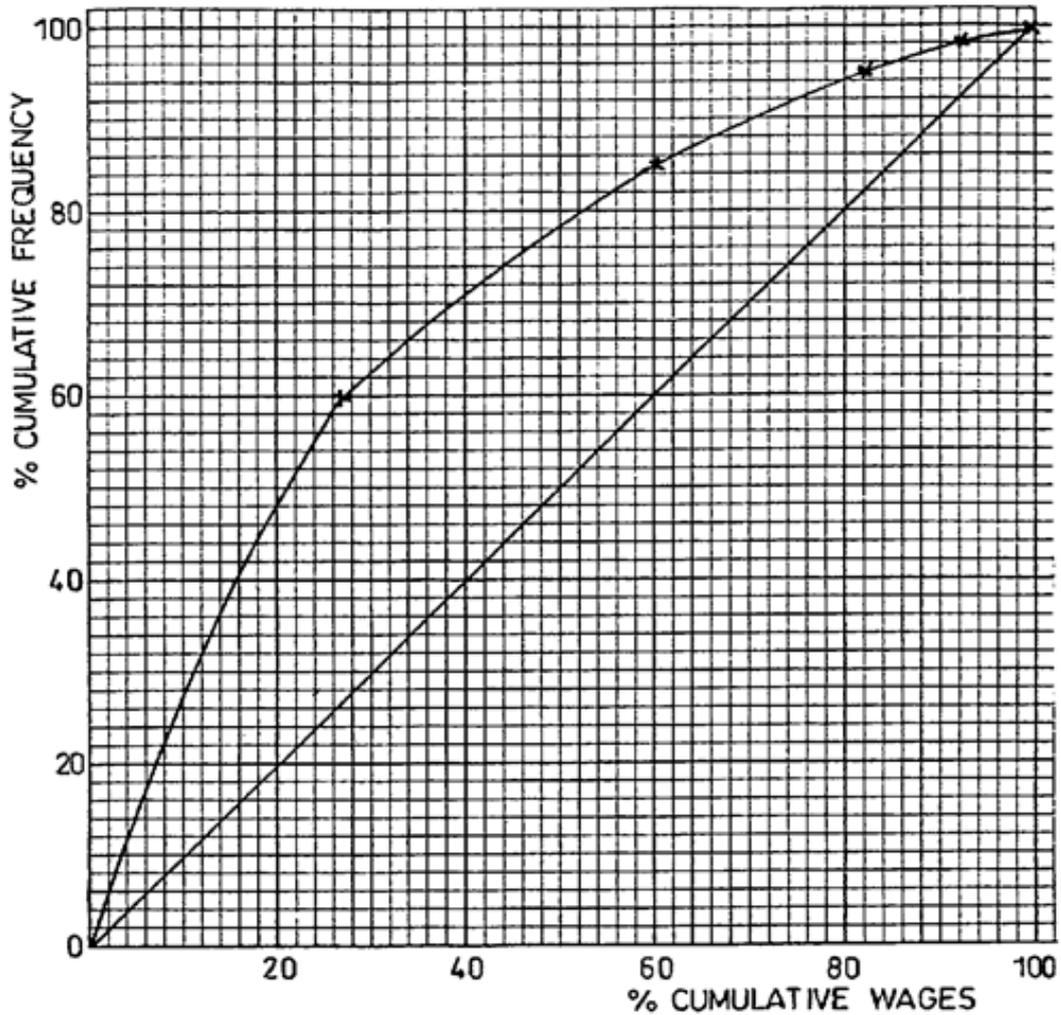


Figure 4.20: Lorenz Curve

Interpretation of the Curve

From Figure 4.20 we can read directly the share of the wages paid to any given percentage of employees:

- a) 50% of the employees earn 22% of the total wages, so we can deduce that the other 50%, i.e. the more highly paid employees, earn 78% of the total wages.
- b) 90% of the employees earn 70% of the total wages, so 10% of the employees must earn 30% of the total wages.
- c) 95% of the employees earn 83% of the total wages, so 5% of the employees earn 17% of the total wages.

Other Uses

Although usually used to show the concentration of wealth (incomes, property ownership, etc.), Lorenz curves can also be employed to show concentration of any other feature. For example, the largest proportion of a country's output of a particular commodity may be produced by only a small proportion of the total number of factories, and this fact can be illustrated by a Lorenz curve.

Concentration of wealth or productivity, etc. may become more or less as time goes on. A series of Lorenz curves on one graph will show up such a state of affairs. In some countries, in recent years, there has been a tendency for incomes to be more equally distributed. A Lorenz curve reveals this because the curves for successive years lie nearer to the straight diagonal.

STUDY UNIT 5

Averages or Measures of Location

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A. THE NEED FOR MEASURES OF LOCATION

We looked at frequency distributions in detail in the previous study unit and you should, by means of a quick revision, make sure that you have understood them before proceeding.

A frequency distribution may be used to give us concise information about its variate, but more often, we will wish to compare two or more distributions. Consider, for example, the distribution of the weights of eggs from two different breeds of poultry (which is a topic in which you would be interested if you were the statistician in an egg marketing company). Having weighed a large number of eggs from each breed, we would have compiled frequency distributions and graphed the results. The two frequency polygons might well look something like Figure 5.1.

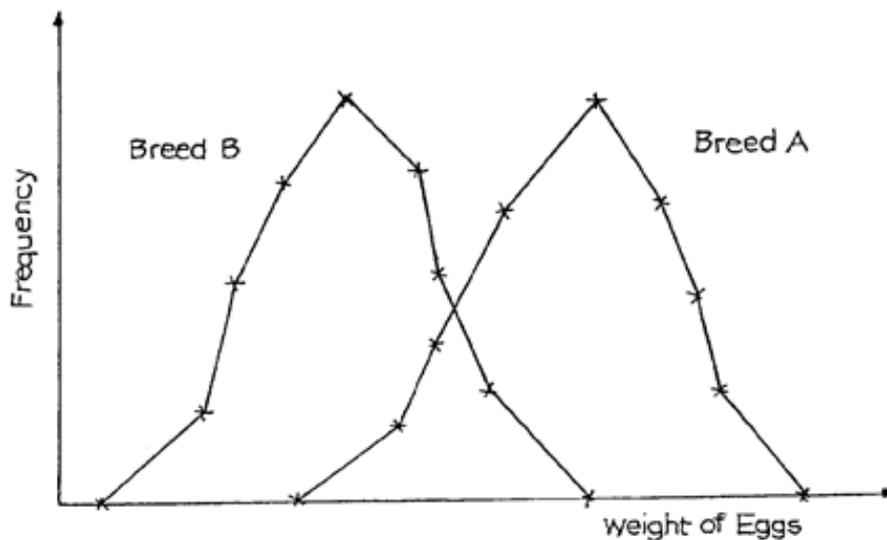


Figure 5.1

Examining these distributions you will see that they look alike except for one thing - they are located on different parts of the scale. In this case the distributions overlap and, although some eggs from Breed A are of less weight than some eggs from Breed B, eggs from Breed A are, in general, heavier than those from Breed B.

Remember that one of the objects of statistical analysis is to condense unwieldy data so as to make it more readily understood. The drawing of frequency curves has enabled us to make an important general statement concerning the relative egg weights of the two breeds of poultry, but we would now like to take the matter further and calculate some figure which will serve to indicate the general level of the variable under discussion. In everyday life we commonly use such a figure when we talk about the "average" value of something or other. We might have said, in reference to the two kinds of egg, that those from Breed A had a higher average weight than those from Breed B. Distributions with different averages indicate that there is a different general level of the variate in the two groups. The single value which we use to describe the general level of the variate is called a "**measure of location**" or a "**measure of central tendency**" or, more commonly, an **average**.

There are three such measures with which you need to be familiar:

- The arithmetic mean
- The mode
- The median.

B. THE ARITHMETIC MEAN

Introduction

This is what we normally think of as the "average" of a set of values. It is obtained by adding together all the values and then dividing the total by the number of values involved. Take, for example, the following set of values which are the heights, in inches, of seven men:

Man	Height (ins)
A	74
B	63
C	64
D	71
E	71
F	66
G	<u>74</u>
Total	<u>483</u>

The arithmetic mean of these heights is $483 \div 7 = 69$ ins. Notice that some values occur more than once, but we still add them all.

At this point we must introduce a little algebra. We don't always want to specify what particular items we are discussing (heights, egg weights, wages, etc.) and so, for general discussion, we use, as you will recall from algebra, some general letter, usually x . Also, we indicate the sum of a number of x 's by Σ (sigma).

Thus, in our example, we may write:

$$\Sigma x = 483$$

We indicate the arithmetic mean by the symbol \bar{x} (called "x bar") and the number of items by the letter n. The calculation of the arithmetic mean can be described by formula thus:

$$\bar{x} = \frac{\sum x}{n}$$

or

$$\bar{x} = \frac{1}{n} \sum x$$

The last one is customary in statistical work. Applying it to the example above, we have:

$$\bar{x} = \frac{1}{n}(483) = 69ins$$

You will often find the arithmetic mean simply referred to as "the mean" when there is no chance of confusion with other means (which we are not concerned with here).

The Mean of a Simple Frequency Distribution

When there are many items (i.e. when n is large) the arithmetic can be eased somewhat by forming a frequency distribution, like this:

Table 5.1

Height in inches (x)	No. of men at this height (f)	Product (fx)
63	1	63
64	1	64
66	1	66
71	2	142
74	2	148
Total	$\Sigma f = 7$	$\Sigma(fx) = 483$

Indicating the frequency of each value by the letter f , you can see that $\sum f = n$ and that, when the x 's are not all the separate values but only the different ones, the formula becomes:

$$\bar{x} = \frac{\sum (fx)}{\sum f}, \text{ which is } 483/7 = 69 \text{ ins as before.}$$

Of course, with only seven items it would not be necessary, in practice, to use this method, but if we had a much larger number of items the method would save a lot of additions.

QUESTION FOR PRACTICE

- a) Consider now Table 5.2. Complete the (fx) column and calculate the value of the arithmetic mean, \bar{x} .

Table 5.2

Value of x	Number of items (f)	Product (fx)
3	1	3
4	4	16
5	7	35
6	15	
7	35	
8	24	
9	8	
10	6	
Total		

**DO NOT READ ON UNTIL YOU HAVE ATTEMPTED TO COMPLETE
AND TO CALCULATE THE ARITHMETIC MEAN.**

You should have obtained the following answers:

The total number of items, $\sum f = 100$

The total product, $\sum (fx) = 713$

The arithmetic mean, $\bar{x} = \frac{713}{100} = 7.13$

Make sure that you understand this study unit so far. Revise it if necessary, before going on to the next paragraph. It is most important that you do not get muddled about calculating arithmetic means.

The Mean of a Grouped Frequency Distribution

Suppose now that you have a grouped frequency distribution. In this case, you will remember, we do not know the actual individual values, only the groups in which they lie. How, then, can we calculate the arithmetic mean? The answer is that we cannot calculate the exact value of \bar{x} , but we can make an approximation sufficiently accurate for most statistical purposes. We do this by assuming that all the values in any group are equal to the mid-point of that group.

The procedure is very similar to that for a simple frequency distribution (which is why I stressed the need for revision) and is shown in this example:

Table 5.3

Group	Mid-Value (x)	Frequency (f)	Product (fx)
0 < 10	5	3	15
10 < 20	15	6	90
20 < 30	25	10	250
30 < 40	35	16	560
40 < 50	45	9	405
50 < 60	55	5	275
60 < 70	65	1	65
Total		$\Sigma f = 50$	$\Sigma (fx) = 1,660$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{1}{50} \times 1,660 = 33.2$$

Provided that Σf is not less than about 50 and that the number of groups is not less than about 12, the arithmetic mean thus calculated is sufficiently accurate for all practical purposes. There is one pitfall to be avoided when using this method; if all the groups should not have the same class interval, be sure that you get the correct mid-values! The following is part of a table with varying class intervals, to illustrate the point:

Group	Mid-Value (\bar{x})
0 < 10	5
10 < 20	15
20 < 40	30
40 < 60	50
60 < 100	80

Table 5.4

You will remember that in discussing the drawing of histograms we had to deal with the case where the last group was not exactly specified. The same rules for drawing the histogram apply to the calculation of the arithmetic mean.

Simplified Calculation

It is possible to simplify the arithmetic still further by the following two devices:

- a) Work from an assumed mean in the middle of one convenient class.
- b) Work in class intervals instead of in the original units.

Let us consider device (a). If you go back to our earlier examples you will discover after some arithmetic that if you add up the differences in value between each reading and the true mean, then these differences add up to zero.

Take first the height distribution discussed at the start of Section B:

$$\bar{x} = 69 \text{ ins}$$

Table 5.5

Man	Height, x , (ins)	$(x - \bar{x})$ (ins)
A	74	5
B	63	-6
C	64	-5
D	71	2
E	71	2
F	66	-3
G	74	5
		$\Sigma(x - \bar{x}) = +14 - 14 = 0$

i.e. $\Sigma(x - \bar{x}) = 0$

Secondly, consider the grouped frequency distribution given earlier in this section:

$$\bar{x} = 33.2$$

Table 5.6

Group	Mid-Value (x)	Frequency (f)	$(x - \bar{x})$	$f(x - \bar{x})$
0 < 10	5	3	-28.2	-84.6
10 < 20	15	6	-18.2	-109.2
20 < 30	25	10	-8.2	-82
30 < 40	35	16	1.8	28.8
40 < 50	45	9	11.8	106.2
50 < 60	55	5	21.8	109
60 < 70	65	1	31.8	31.8
Totals	$\Sigma f = 50$		$\Sigma(f(x - \bar{x})) = +275.8 - 275.8 = 0$	

i.e. $\Sigma f(x - \bar{x}) = 0$

If we take any value other than \bar{x} and follow the same procedure, the sum of the differences (sometimes called deviations) will not be zero. In our first example, let us assume the mean to be 68 ins and label the assumed mean x_0 . The differences between each reading and this assumed value are:

Table 5.7

Man	Height, x , (ins)	$d = (x - x_0)$ (ins)
A	74	6
B	63	-5
C	64	-4
D	71	3
E	71	3
F	66	-2
G	74	6
		$\Sigma(x - x_0) = +18 - 11 = 7$

i.e. $\Sigma(x - x_0) = +7$ ins or $\Sigma d = +7$ ins

We make use of this property and we use this method as a "short-cut" for finding \bar{x} . Firstly, we have to choose some value of x as an **assumed mean**. We try to choose it near to where we think the true mean, x , will lie, and we always choose it as the mid-point of one of the groups when we are involved with a grouped frequency distribution. In the above example, the total deviation, d , does not equal zero, so 68 cannot be the true mean. As the total deviation is positive, we must have UNDERESTIMATED in our choice of x_0 , so the true mean is higher than 68. As there are seven readings, we need to adjust x_0 upwards by one seventh of the total deviation, i.e. by $(+7)/7 = +1$. Therefore the true value of \bar{x} is:

$$68 + \frac{(+7)}{7} = 68 + 1 = 69\text{ins}$$

We know this to be the correct answer from our earlier work.

Let us now illustrate the "short-cut" method for the grouped frequency distribution. We shall take x_0 as 35 as this is the mid-value in the centre of the distribution.

Table 5.8

Group	Mid-Value (x)	Frequency (f)	d = (x - x _o)	f(x - x _o) = fd
0 < 10	5	3	-30	- 90
10 < 20	15	6	-20	- 120
20 < 30	25	10	-10	- 100
30 < 40	35	16	0	0
40 < 50	45	9	10	90
50 < 60	55	5	20	100
60 < 70	65	1	30	30
Total	Σf = 50		Σfd = -310 + 220 = -90	

$$\Sigma fd = -90$$

This time we must have OVERESTIMATED x_o , as the total deviation, Σfd , is negative. As there are 50 readings altogether, the true mean must be $\frac{1}{50}$ th of the (-90) lower than 35, i.e.

$$\bar{x} = 35 = \frac{(-90)}{50} = 35 - 1.8 = 33.2$$

which is as we found previously.

Device (b) can be used with a grouped frequency distribution to work in units of the class interval instead of in the original units. In the fourth column of Table 43, you can see that all the deviations are multiples of 10, so we could have worked in units of 10 throughout and then compensated for this at the end of the calculation.

Let us repeat the calculation using this method. The result (with $x_o = 35$) is:

Table 5.9

Group	Mid-Value (x)	Frequency (f)	(x - x ₀)	$d = \frac{x - x_0}{c}$	fd
0 < 10	5	3	-30	-3	-9
10 < 20	15	6	-20	-2	-12
20 < 30	25	10	-10	-1	-10
30 < 40	35	16	0	0	0
40 < 50	45	9	10	1	9
50 < 60	55	5	20	2	10
60 < 70	65	1	30	3	3
Totals	$\Sigma f = 50$			$\Sigma \frac{x - x_0}{c} = +22 - 31 = -9$	

The symbol used for the length of the class interval is c, but you may also come across the symbol i used for this purpose.

The mean $\bar{x} = 35 + \frac{(-9)}{50} \times 10$, because we multiply by 10 at this stage to compensate for working in class interval units in the table.

Thus $\bar{x} = 35 - \frac{90}{50} = 35 - 1.8 = 33.2$ as before.

The general formula for \bar{x} that applies for all grouped frequency distributions having equal class intervals can be written as

$\bar{x} = x_0 + \frac{\sum fd}{n} \times c$	where d is now	$\frac{x - x_0}{c}$
--	----------------	---------------------

As we mentioned at an earlier stage, you have to be very careful if the class intervals are unequal, because you can only use one such interval as your working unit. Table 5.10 shows you how to deal with this situation.

Table 5.10

Group	Mid-Value (x)	$d = \frac{x - x_0}{c}$	Frequency (f)	Product (fd)
0 < 10	5	-3	3	-9
10 < 20	15	-2	6	-12
20 < 30	25	-1	10	-10
30 < 40	35	0	16	0
40 < 50	45	+1	9	+9
50 < 70	60	+21	6	+15
Total			50	+24) = -7 -31)

The assumed mean is 35, as before, and the working unit is a class interval of 10. Notice how d for the last group is worked out; the mid-point is 60, which is $2^{1/2}$ times 10 above the assumed mean. The required arithmetic mean is, therefore:

$$\bar{x} = 35 - \frac{7 \times 10}{50} = 35 - \frac{70}{50} = 35 - 1.4 = 33.6$$

We have reached a slightly different figure from before because of the error introduced by the coarser grouping in the "50-70" region.

The method just described is of great importance both in work day statistics and in examinations. By using it correctly, you can often do the calculations for very complicated-looking distributions by using mental arithmetic and pencil and paper.

With the advent of electronic calculators, the time saving on calculations of the arithmetic mean is not great, but this method is still preferable because:

- The numbers involved are smaller and thus you are less likely to make slips in arithmetic.
- The method can be extended to enable us to find easily the standard deviation of a frequency distribution.

Characteristics of the Arithmetic Mean

There are a number of characteristics of the arithmetic mean which you must know and understand. Apart from helping you to understand the topic more thoroughly, the following are the points which an examiner expects to see when he or she asks for "brief notes" on the arithmetic mean:

- a) It is not necessary to know the value of every item in order to calculate the arithmetic mean. Only the total and the number of items are needed. For example, if you know the total wages bill and the number of employees, you can calculate the arithmetic mean wage without knowing the wages of each person.
- b) It is fully representative because it is based on all, and not only some, of the items in the distribution.
- c) One or two extreme values can make the arithmetic mean somewhat unreal by their influence on it. For example, if a millionaire came to live in a country village, the inclusion of his income in the arithmetic mean for the village would make the place seem very much better off than it really was!
- d) The arithmetic mean is reasonably easy to calculate and to understand.
- e) In more advanced statistical work it has the advantage of being amenable to algebraic manipulation.

QUESTION FOR PRACTICE

- 1) Table 5.11 shows the consumption of electricity of 100 householders during a particular week. Calculate the arithmetic mean consumption of the 100 householders.

Table 5.11

Consumption (kilowatt hours)	Number of Householders
0 - under 10	5
10 - " 20	8
20 - " 30	20
30 - " 40	29
40 - " 50	20
50 - " 60	11
60 - " 70	6
70 - " 80	1
80 or over	<u>0</u>
	<u>100</u>

C. THE MODE

Mode of a Simple Frequency Distribution

The first alternative to the mean which we will discuss is the mode. This is the name given to the most frequently occurring value. Look at the following frequency distribution:

Table 5.12

Number of accidents per day (x)	Number of days with the stated number of accidents (f)
0	27
1	39
2	30
3	20
4	7
Total	123

In this case the most frequently occurring value is 1 (it occurred 39 times) and so the mode of this distribution is 1. Note that the mode, like the mean, is a value of the variate, x , not the frequency of that value. A common error is to say that the mode of the above distribution is 39. **THIS IS WRONG.** The mode is 1. Watch out, and do not fall into this trap!

For comparison, calculate the arithmetic mean of the distribution: it works out at 1.52. The mode is used in those cases where it is essential for the measure of location to be an actually occurring value. An example is the case of a survey carried out by a clothing store to determine what size of garment to stock in the greatest quantity. Now, the average size of garment in demand might turn out to be, let us say, 9.3724, which is not an actually occurring value and doesn't help us to answer our problem. However, the mode of the distribution obtained from the survey would be an actual value (perhaps size 8) and it would provide the answer to the problem.

Mode of a Grouped Frequency Distribution

When the data is given in the form of a **grouped frequency distribution**, it is not quite so easy to determine the mode. What, you might ask, is the mode of the following distribution?

Table 5.13

Group	Frequency
0 < 10	4
10 < 20	6
20 < 30	10
30 < 40	16
40 < 50	24
50 < 60	32
60 < 70	38
70 < 80	40
80 < 90	20
90 < 100	10
100 < 110	5
110 < 120	1

All we can really say is that "70-80" is the **modal group** (the group with the largest frequency). You may be tempted to say that the mode is 75, but this is not true, nor even a useful approximation in most cases. The reason is that the modal group depends on the method of grouping, which can be chosen quite arbitrarily to suit our convenience. The distribution could have been set out with class intervals of five instead of 10, and would then have appeared as follows (only the middle part is shown, to illustrate the point):

Table 5.14

Group	Frequency
60 - 65	16
65 - 70	22
70 - 75	21
75 - 80	19
80 - 85	12
85 - 90	8

The modal group is now "65-70". Likewise, we will get different modal groups if the grouping is by 15 or by 20 or by any other class interval, and so the mid-point of the modal group is not a good way of estimating the mode.

In practical work, this determination of the modal group is usually sufficient, but examination papers occasionally ask for the mode to be determined from a grouped distribution.

A number of procedures based on the frequencies in the groups adjacent to the modal group can be used, and I will now describe one procedure. You should note, however, that these procedures are only mathematical devices for finding the MOST LIKELY position of the mode; it is not possible to calculate an exact and true value in a grouped distribution.

We saw that the modal group of our original distribution was "70-80". Now examine the groups on each side of the modal group; the group below (i.e. 60-70) has a frequency of 38, and the one above (i.e. 80-90) has a frequency of 20. This suggests to us that the mode may be some way towards the lower end of the modal group rather than at the centre. A graphical method for estimating the mode is shown in Figure 5.2.

This method can be used when the distribution has equal class intervals. Draw that part of the histogram which covers the modal class and the adjacent classes on either side.

Draw in the diagonals AB and CD as shown in Figure 5.2. From the point of intersection draw a vertical line downwards. Where this line crosses the horizontal axis is the mode. In our example the mode is just less than 71.

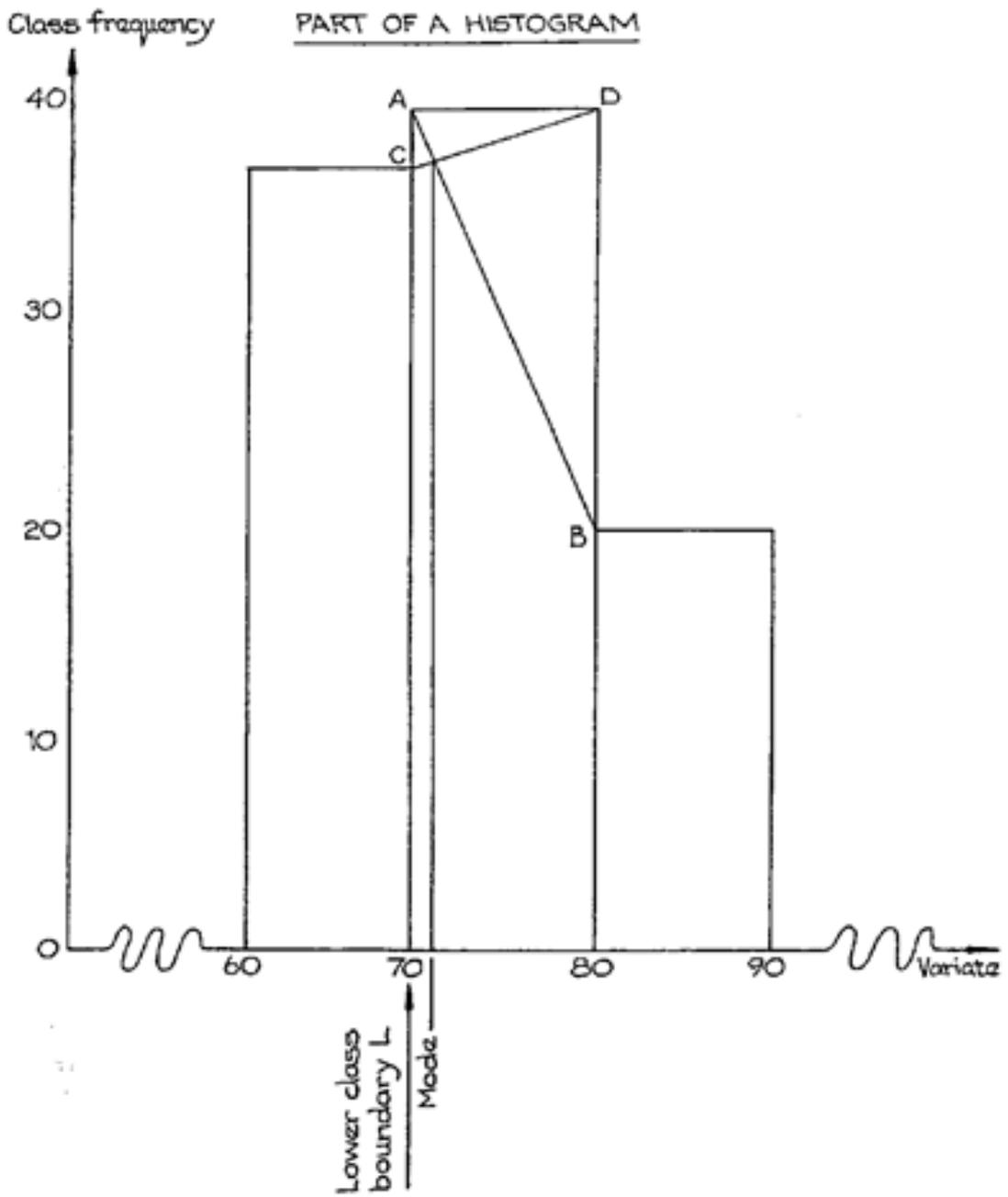


Figure 5.2

Characteristics of the Mode

Some of the characteristics of the mode are worth noting as you may well be asked to compare them with those of the arithmetic mean.

- a) The mode is very easy to find with ungrouped distributions, since no calculation is required.
- b) It can only be determined roughly with grouped distributions.
- c) It is not affected by the occurrence of extreme values.
- d) Unlike the arithmetic mean, it is not based on all the items in the distribution, but only on those near its value.
- e) In ungrouped distributions the mode is an actually occurring value.
- f) It is not amenable to the algebraic manipulation needed in advanced statistical work.
- g) It is not unique, i.e. there can be more than one mode. For example, in the set of numbers, 6, 7, 7, 7, 8, 8, 9, 10, 10, 10, 12, 13, there are two modes, namely 7 and 10. This set of numbers would be referred to as having a **bimodal** distribution.
- h) The mode may not exist. For example, in the set of numbers 7, 8, 10, 11, 12, each number occurs only once so this distribution has no mode.

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D. THE MEDIAN

Introduction

The desirable feature of any measure of location is that it should be near the middle of the distribution to which it refers. Now, if a value is near the middle of the distribution, then we expect about half of the distribution to have larger values, and the other half to have smaller values. This suggests to us that a possible measure of location might be that value which is such that exactly half (i.e. 50%) of the distribution has larger values and exactly half has lower values. The value which so divides the distribution into equal parts is called the MEDIAN. Look at the following set of values:

6, 7, 7, 8, 8, 9, 10, 10, 10, 12, 13

The total of these eleven numbers is 100 and the arithmetic mean is therefore $100/11 = 9.091$, while the mode is 10 because that is the number which occurs most often (three times). The median, however, is 9 because there are five values above and five values below 9. Our first rule for determining the median is therefore as follows:

Arrange all the values in order of magnitude and the median is then the middle value.

Note that all the values are to be used: even though some of them may be repeated, they must all be put separately into the list. In the example just dealt with, it was easy to pick out the middle value because there was an odd number of values. But what if there is an even number? Then, by convention, the median is taken to be the arithmetic mean of the two values in the middle. For example, take the following set of values:

6, 7, 7, 8, 8, 9, 10, 10, 11, 12

The two values in the middle are 8 and 9, so that the median is 8.5

Median of a Simple Frequency Distribution

Statistical data, of course, is rarely in such small groups and, as you have already learned, we usually deal with frequency distributions. How, then do we find the median if our data is in the form of a distribution?

Let us take the example of the frequency distribution of accidents already used in discussing the mode. The total number of values is 123 and so when those values are arranged in order of magnitude, the median will be the 62nd item because that will be the middle item. To see what the value of the 62nd item will be, let us again draw up the distribution:

Table 5.15

Number of accidents per day (x)	Number of days with the stated number of accidents (f)	Cumulative frequency (F)
0	27	27
1	39	66
2	30	96
3	20	116
4	7	123
Total	123	

You can see from the last column that, if we were to list all the separate values in order, the first 27 would all be 0s and from then up to the 66th would be 1s; it follows therefore that the 62nd item would be a 1 and that the median of this distribution is 1.

Median of a Grouped Frequency Distribution

The final problem connected with the median is how to find it when our data is in the form of a grouped distribution. The solution to the problem, as you might expect, is very similar to the solution for an ungrouped distribution; we halve the total frequency and then find, from the cumulative frequency column, the corresponding value of the variate.

Because a grouped frequency distribution nearly always has a large total frequency, and because we do not know the exact values of the items in each group, it is not necessary to find the two middle items when the total frequency is even: just halve the total frequency and use the answer (whether it is a whole number or not) for the subsequent calculation.

Table 5.16

Group	Frequency (f)	Cumulative Frequency (F)
0 < 10	4	4
10 < 20	6	10
20 < 30	10	20
30 < 40	16	36
40 < 50	24	60
50 < 60	32	92
60 < 70	38	130
70 < 80	40	170
80 < 90	20	190
90 < 100	10	200
100 < 110	5	205
110 < 120	1	206
Total	206	

The total frequency is 206 and therefore the median is the 103rd item which, from the cumulative frequency column, must lie in the 60-70 group. But exactly where in the 60-70 group? Well, there are 92 items before we get to that group and we need the 103rd item, so we obviously need to move into that group by 11 items. Altogether in our 60-70 group there are 38 items so we need to move 11/38 of the way into that group, that is 11/38 of 10 above 60. Our median is therefore

$$60 + 110/38 = 60 + 2.89 = 62.89.$$

The use of the cumulative frequency distribution will, no doubt, remind you of its graphical representation, the ogive. In practice, a convenient way to find the median of a grouped distribution is to draw the ogive and then, against a cumulative frequency of half the total frequency, to read off the median. In our example the median would be read against 103 on the cumulative frequency scale (see Figure 5.3). If the ogive is drawn with relative frequencies, then the median is always read off against 50%.

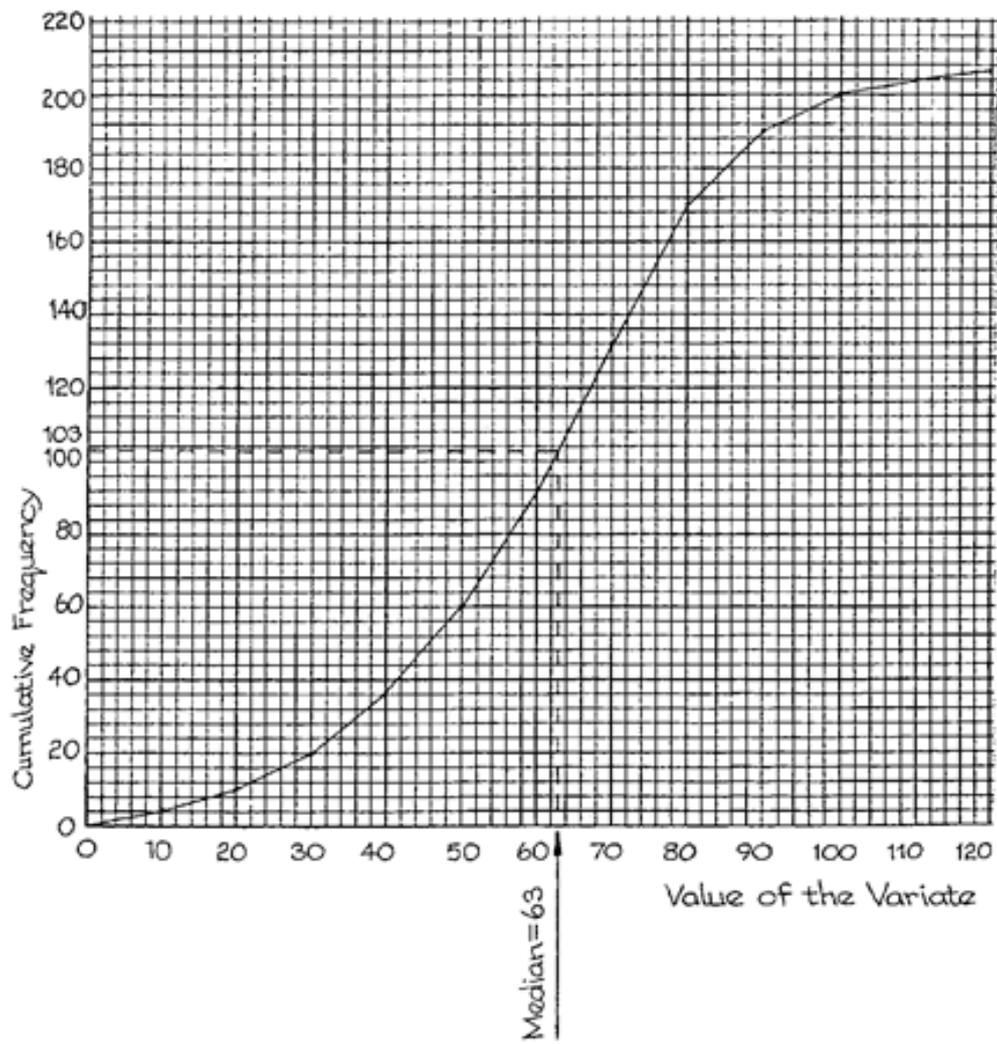


Figure 5.3

Characteristics of the Median

Characteristic features of the median, which you should compare with those of the mean and the mode, are as follows:

- a) It is fairly easily obtained in most cases, and is readily understood as being the "half-way point".
- b) It is less affected by extreme values than the mean. The millionaire in the country village might alter considerably the mean income of the village but he would have almost no effect at all on the median.
- c) It can be obtained without actually having all the values. If, for example, we want to know the median height of a group of 21 men, we do not have to measure the height of every single one; it is only necessary to stand the men in order of their heights and then only the middle one (No. 11) need be measured, for his height will be the median height. The median is thus of value when we have open-ended classes at the edges of the distribution as its calculation does not depend on the precise values of the variate in these classes, whereas the value of the arithmetic mean does.
- d) The median is not very amenable to further algebraic manipulation.

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STUDY UNIT 6

Measures of Dispersion

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A. INTRODUCTION TO DISPERSION

In order to get an idea of the general level of values in a frequency distribution, we have studied the various measures of location that are available. However, the figures which go to make up a distribution may all be very close to the central value, or they may be widely dispersed about it, e.g. the mean of 49 and 51 is 50, but the mean of 0 and 100 is also 50! You can see, therefore, that two distributions may have the same mean but the individual values may be spread about the mean in vastly different ways.

When applying statistical methods to practical problems, a knowledge of this spread (which we call "dispersion" or "variation") is of great importance. Examine the figures in the following table:

Table 6.1

Week	Weekly Output	
	Factory A	Factory B
1	94	136
2	100	92
3	106	110
4	100	36
5	90	102
6	101	57
7	107	108
8	98	81
9	101	156
10	98	117
Total	995	995
Mean Output	99.5	99.5

Although the two factories have the same mean output, they are very different in their week-to-week consistency. Factory A achieves its mean production with only very little variation from week to week, whereas Factory B achieves the same mean by erratic ups-and-downs from week to week. This example shows that a mean (or other measure of location) does not, by itself, tell the whole story and we therefore need to supplement it with a "measure of dispersion".

As was the case with measures of location, there are several different measures of dispersion in use by statisticians. Each has its own particular merits and demerits, which will be discussed later. The measures in common use are:

- Range
- Quartile deviation
- Mean deviation
- Standard deviation
-

We will discuss three of these here.

B. THE RANGE

This is the simplest measure of dispersion; it is simply the difference between the largest and the smallest. In the example just given, we can see that the lowest weekly output for Factory A was 90 and the highest was 107; the range is therefore 17. For Factory B the range is $156 - 36 = 120$. The larger range for Factory B shows that it performs less consistently than Factory A.

The advantage of the range as a measure of the dispersion of a distribution is that it is very easy to calculate and its meaning is easy to understand. For these reasons it is used a great deal in industrial quality control work. Its disadvantage is that it is based on only two of the individual values and takes no account of all those in between. As a result, one or two extreme results can make it quite unrepresentative. Consequently, the range is not much used except in the case just mentioned.

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C. THE QUARTILE DEVIATION, DECILES AND PERCENTILES

The Quartile Deviation

This measure of dispersion is sometimes called the "semi-interquartile range". To understand it, you must cast your mind back to the method of obtaining the median from the ogive. The median, you remember, is the value which divides the total frequency into two halves. The values which divide the total frequency into quarters are called **quartiles** and they can also be found from the ogive, as shown in Figure 6.1.

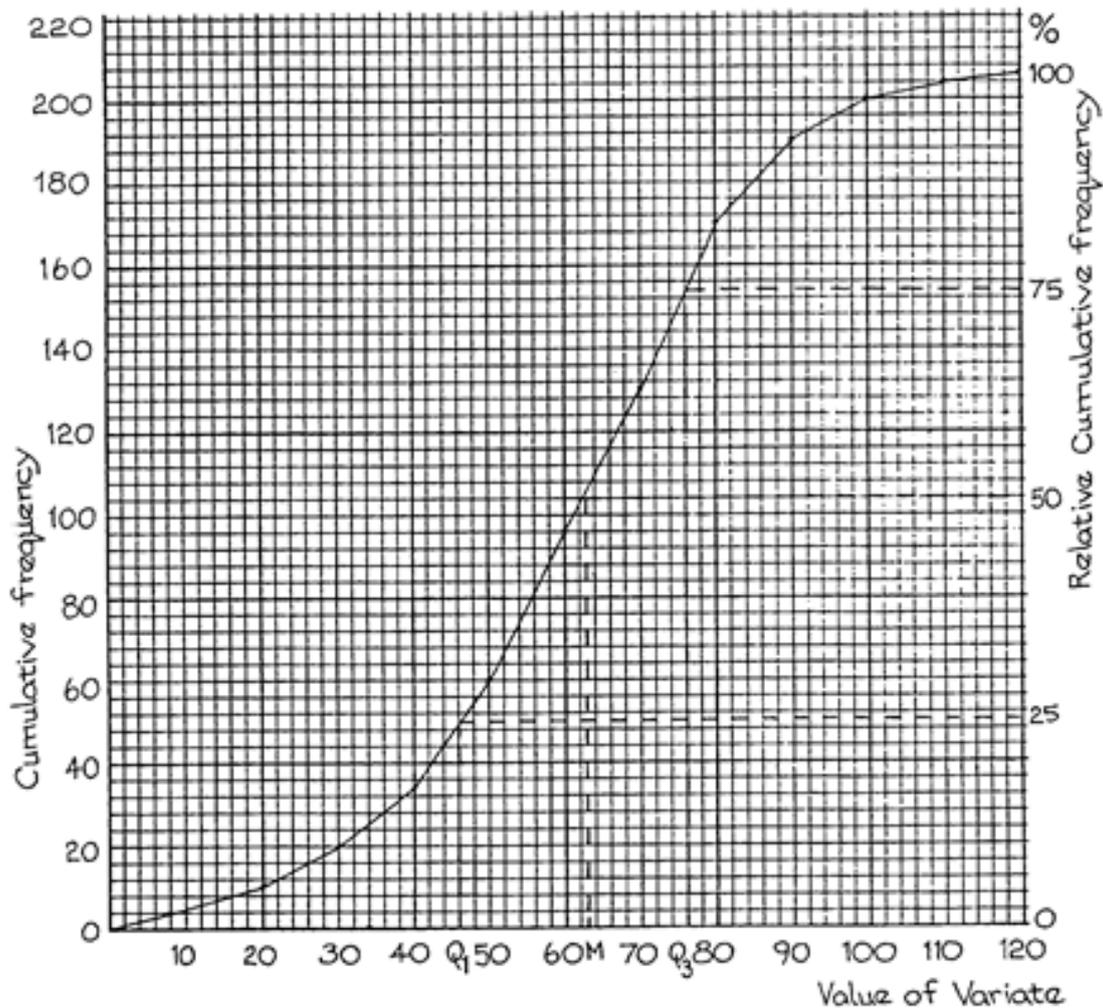


Figure 6.1

This is the same ogive that we drew earlier when finding the median of the grouped frequency distribution featured in Section D of the previous study unit.

You will notice that we have added the relative cumulative frequency scale to the right of the graph. 100% corresponds to 206, i.e. the total frequency. It is then easy to read off the values of the variate corresponding to 25%, 50% and 75% of the cumulative frequency, giving the lower quartile (Q_1), the median and the upper quartile (Q_3) respectively.

$$Q_1 = 46.5$$

$$\text{Median} = 63 \text{ (as found previously)}$$

$$Q_3 = 76$$

The difference between the two quartiles is the **interquartile range** and half of the difference is the **semi-interquartile range or quartile deviation**:

$$\begin{aligned} \text{i.e. quartile deviation} &= \frac{Q_3 - Q_1}{2} \\ &= \frac{76 - 46.5}{2} \\ &= \frac{29.5}{2} \\ &= 14.75 \end{aligned}$$

Alternatively, you can work out 25% of the total frequency, i.e. $\frac{206}{4} = 51.5$ and 75% of

the total frequency, i.e. 154.5, and read from the ogive the values of the variate corresponding to 51.5 and 154.5 on the cumulative frequency scale (i.e. the left-hand scale). The end result is the same.

Calculation of the Quartile Deviation

The quartile deviation is not difficult to calculate and some examination questions may specifically ask for it to be calculated, in which case a graphical method is not acceptable. Graphical methods are never quite as accurate as calculations.

We shall again use the same example.

The table of values is reproduced for convenience:

Table 6.2

Group	Frequency (f)	Cumulative Frequency (F)
0 < 10	4	4
10 < 20	6	10
20 < 30	10	20
30 < 40	16	36
40 < 50	24	60
50 < 60	32	92
60 < 70	38	130
70 < 80	40	170
80 < 90	20	190
90 < 100	10	200
100 < 110	5	205
110 < 120	1	206
Total	206	

The lower quartile is the 25% point in the total distribution and the upper quartile the 75% point.

As the total frequency is 206, the 25% point is: $\frac{206}{4} = 51\frac{1}{2}$

and the 75% point is $\frac{3}{4} \times 206 = 154\frac{1}{2}$.

We can make the calculations in exactly the same manner as we used for calculating the median - we saw this in Section D of the previous study unit.

Looking at Table 6.2, the 51½th item comes in the 40-50 group and will be the (51½ – 36) = 15½th item within it.

$$\begin{aligned}
 \text{So, the lower quartile} &= 40 + \frac{15 \frac{1}{2}}{24} \times 10 \\
 &= 40 + 6.458 \\
 &= \mathbf{46.458}
 \end{aligned}$$

Similarly, the upper quartile will be the 154th item which is in the 70-80 group and is the (154 – 130) = 24th item within it.

$$\begin{aligned}
 \text{So, the upper quartile} &= 70 + \frac{24 \frac{1}{2}}{40} \times 10 \\
 &= 70 + 6.125 \\
 &= \mathbf{76.125}
 \end{aligned}$$

Remember that the units of the quartiles and of the median are the same as those of the variate.

$$\begin{aligned}
 \text{The quartile deviation} &= \frac{Q_3 - Q_1}{2} \\
 &= \frac{76.125 - 46.458}{2} \\
 &= \frac{29.667}{2} \\
 &= 14.8335 \\
 &= \mathbf{14.8 \text{ to 1 decimal place}}
 \end{aligned}$$

The quartile deviation is unaffected by an occasional extreme value. It is not based, however, on the actual value of all the items in the distribution and to this extent it is less representative than the standard deviation. In general, when a median is the appropriate measure of location then the quartile deviation should be used as the measure of dispersion.

Deciles and Percentiles

It is sometimes convenient, particularly when dealing with wages and employment statistics, to consider values similar to the quartiles but which divide the distribution more finely. Such partition values are deciles and percentiles. From their names you will probably have guessed that the deciles are the values which divide the total frequency into tenths and the percentiles are the values which divide the total frequency into hundredths. Obviously it is only meaningful to consider such values when we have a large total frequency.

The deciles are labelled $D_1, D_2 \dots D_9$: the second decile D_2 , for example, is the value below which 20% of the data lies and the sixth decile D_6 is the value below which 60% of the data lies.

The percentiles are labelled $P_1, P_2 \dots P_{99}$ and, for example, P_5 is the value below which 5% of the data lies and P_{64} is the value below which 64% of the data lies.

Using the same example as above, let us calculate, as an illustration, the third decile D_3 . The method follows exactly the same principles as the calculation of the median and quartiles.

D_3 is the value which 30% of the data lies. 30% or 206 is $\frac{30}{100} \times 206 = 61.8$

so we are looking for the value of the 61.8th item. A glance at the cumulative frequency column shows that the 61.8th item lies in the 50-60 group, and is the $(61.8 - 60) = 1.8$ th item within it.

So,

$$\begin{aligned} D_3 &= 50 + \frac{1.8}{32} \times 10 \\ &= 50 + \frac{18}{32} \\ &= \mathbf{50.6} \text{ to 1 dec. place} \end{aligned}$$

Therefore 30% of our data lies below 50.6.

We could also have found this result graphically; again check that you agree with the calculation by reading D_3 from the graph. You will see that the calculation method enables us to give a more precise answer than is obtainable graphically.

D. THE STANDARD DEVIATION

Most important of the measures of dispersion is the standard deviation. Except for the use of the range in statistical quality control and the use of the quartile deviation in wages statistics, the standard deviation is used almost exclusively in statistical practice. It is defined as the **square root of the variance** and so we need to know how to calculate the variance first.

The Variance

We start by finding the deviations from the mean, and then squaring them, which removes the negative signs in a mathematically acceptable fashion, thus:

Table 6.3

Week	Weekly Output (x)	Deviation (x - \bar{x})	Deviation Squared (x - \bar{x}) ²
1	94	- 5.5	30.25
2	100	+ 0.5	0.25
3	106	+ 6.5	42.25
4	100	+ 0.5	0.25
5	90	- 9.5	90.25
6	101	+ 1.5	2.25
7	107	+ 7.5	56.25
8	98	- 1.5	2.25
9	101	+ 1.5	2.25
10	98	- 1.5	2.25
Total	995	+ 18.0 - 18.0	228.50
Mean	99.5	Variance = 22.85	

The mean of the squared deviations is the variance. The standard deviation is, therefore,

$$\sqrt{22.85} = 4.78.$$

Standard Deviation of a Simple Frequency Distribution

If the data had been given as a frequency distribution (as is often the case) then only the different values would appear in the "x" column and we would have to remember to multiply each result by its frequency:

Table 6.4

Weekly Output (x)	Frequency (f)	fx	Deviation (x - \bar{x})	Deviation Squared (x - \bar{x}) ²	f(x - \bar{x}) ²
90	1	90	- 9.5	90.25	90.25
94	1	94	- 5.5	30.25	30.25
98	2	196	- 1.5	2.25	4.50
100	2	200	+ 0.5	0.25	0.50
101	2	202	+ 1.5	2.25	4.50
106	1	106	+ 6.5	42.25	42.25
107	1	107	+ 7.5	56.25	56.25
Total	10	995			228.50
Mean		99.5		Variance = 22.85	

The formula for working out the standard deviation SD is:

$$SD = \sqrt{\frac{\sum f(x - \bar{x})^2}{n}}$$

where n is the total frequency.

Standard Deviation of a Grouped Frequency Distribution

When we come to the problem of finding the standard deviation of a grouped frequency distribution, we again assume that all the readings in a given group fall at the mid-point of the group, so we can find the arithmetic mean as before. Let us use the following distribution, with the mean deviation,

$$x = 41.7.$$

Table 6.5

Class	Mid-Value (x)	Frequency (f)	Deviation from the True Mean (x - \bar{x})	(x - \bar{x}) ²	f(x - \bar{x}) ²
10 < 20	15	2	- 26.7	712.89	1,425.78
20 < 30	25	5	- 16.7	278.89	1,394.45
30 < 40	35	8	- 6.7	44.89	359.12
40 < 50	45	6	3.3	10.89	65.34
50 < 60	55	5	13.3	176.89	884.45
60 < 70	65	3	23.3	542.89	1,628.67
70 < 80	75	1	33.3	1,108.89	1,108.89
Total		30			6,866.70
				Variance = 228.89	

$$SD = \sqrt{228.89} = 15.13$$

The arithmetic is rather tedious even with an electronic calculator, but we can extend the "short-cut" method which we used for finding the arithmetic mean of a distribution, to find the standard deviation as well. In that method we:

- Worked from an assumed mean.
- Worked in class intervals.
- Applied a correction to the assumed mean.

Table 6.6 shows you how to work out the standard deviation.

Table 6.6

Class	Mid-Value (x)	Frequency (f)	$d = \frac{x - x_0}{c}$	fd	d^2	fd^2
10 < 20	15	2	-2	-4	4	8
20 < 30	25	5	-1	-5	1	5
30 < 40	35	8	0	0	0	0
40 < 50	45	6	+1	+6	1	6
50 < 60	55	5	+2	+10	4	20
60 < 70	65	3	+3	+9	9	27
70 < 80	75	1	+4	+4	16	16
Total		30		+29) -9) +20		82

$$x_0 = 35, c = 10$$

The standard deviation is calculated in four steps from this table, as follows:

(1) The approximate variance is obtained from $\frac{1}{n} \times \sum fd^2$ which in our case is equal to

$$\frac{1}{30} \times 82 = 82/30 = 2.7333.$$

(2) The correction is $-\left(\frac{1}{n} \times \sum fd\right)^2$ i.e. it is always SUBTRACTED from the approximate variance to get the corrected variance. In our case the correction is $-(20/30)^2 = -0.4444$.

(3) The **corrected variance** is thus:

$$2.7333 - 0.4444 = 2.2889$$

(4) The **standard deviation** is then the class interval times the square root of the corrected variance:

$$\begin{aligned} SD &= \sqrt{2.2889} = 1.513 \text{ class intervals} \times 10 \\ &= \mathbf{15.13} \end{aligned}$$

This may seem a little complicated, but if you work through the example a few times, it will all fall into place. Remember the following points:

- a) Work from an assumed mean at the mid-point of any convenient class.
- b) The correction is always subtracted from the approximate variance.
- c) As you are working in class intervals, it is necessary to multiply by the class interval as the last step.
- d) The correction factor is the same as that used for the "short-cut" calculation of the mean, but for the SD it has to be squared.
- e) The column for d^2 may be omitted since $fd^2 = fd$ multiplied by d . But do not omit it until you have really grasped the principles involved.

f) The formula for calculating the standard deviation in this way can be written as:

$$SD = c \sqrt{\frac{\sum fd^2}{n} - \left(\frac{\sum fd}{n}\right)^2}$$

- g) The assumed mean should be chosen from a group with the most common interval and c will be that interval. If the intervals vary too much, we revert to the basic formula.

Characteristics of the Standard Deviation

In spite of the apparently complicated method of calculation, the standard deviation is the measure of dispersion used in all but the very simplest of statistical studies. It is based on all of the individual items, it gives slightly more emphasis to the larger deviations but does not ignore the smaller ones and, most important, it can be treated mathematically in more advanced statistics.

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E. THE COEFFICIENT OF VARIATION

Suppose that we are comparing the profits earned by two businesses. One of them may be a fairly large business with average monthly profits of RWF50,000, while the other may be a small firm with average monthly profits of only RWF2,000. Clearly, the general level of profits is very different in the two cases, but what about the month-by-month variability? We will compare the two firms as to their variability by calculating the two standard deviations; let us suppose that they both come to RWF500. Now, RWF500 is a much more significant amount in relation to the small firm than it is in relation to the large firm so that, although they have the same standard deviations, it would be unrealistic to say that the two businesses are equally consistent in their month-to-month earnings of profits. To overcome the difficulty, we express the SD as a percentage of the mean in each case and we call the result the "coefficient of variation".

Applying the idea to the figures which we have just quoted, we get coefficients of variation (usually indicated in formulae by V or CV) as follows:

$$(a) \text{ For the large firm, } V = \frac{500}{50,000} \times 100 = 1.0\%$$

$$(b) \text{ For the small firm, } V = \frac{500}{2,000} \times 100 = 25.0\%$$

This shows that, relatively speaking, the small firm is more erratic in its earnings than the large firm.

Note that although a standard deviation has the same units as the variate, the coefficient of variation is a ratio and thus has no units.

Another application of the coefficient of variation comes when we try to compare distributions the data of which are in different units as, for example, when we try to compare a French business with an American business. To avoid the trouble of converting the dollars to euro (or vice versa) we can calculate the coefficients of variation in each case and thus obtain comparable measures of dispersion.

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F. SKEWNESS

When the items in a distribution are dispersed equally on each side of the mean, we say that the distribution is symmetrical. Figure 6.2 shows two symmetrical distributions.

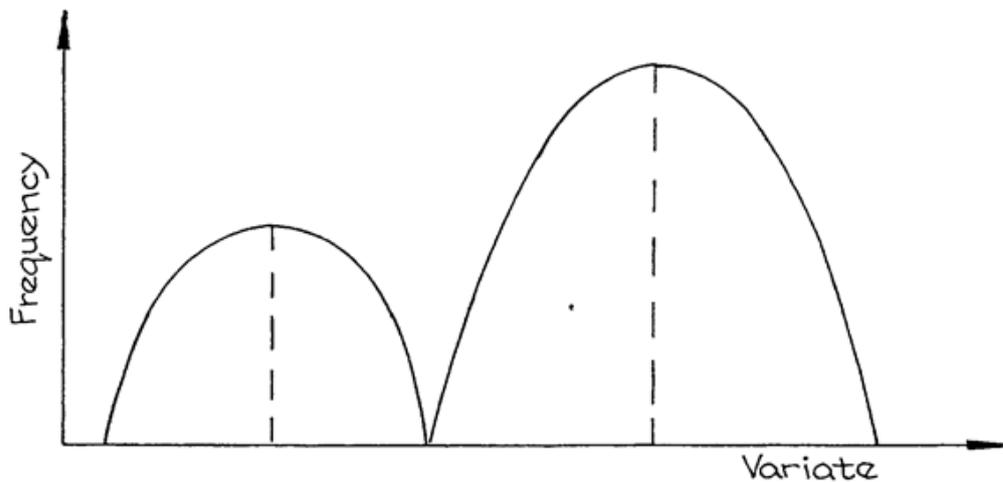


Figure 6.2

When the items are not symmetrically dispersed on each side of the mean, we say that the distribution is **skew** or asymmetric.

A distribution which has a tail drawn out to the right is said to be **positively skew**, while one with a tail to the left, is **negatively skew**. Two distributions may have the same mean and the same standard deviation but they may be differently skewed. This will be obvious if you look at one of the skew distributions in Figure 6.3 and then look at the **same one** through from the other side of the paper!

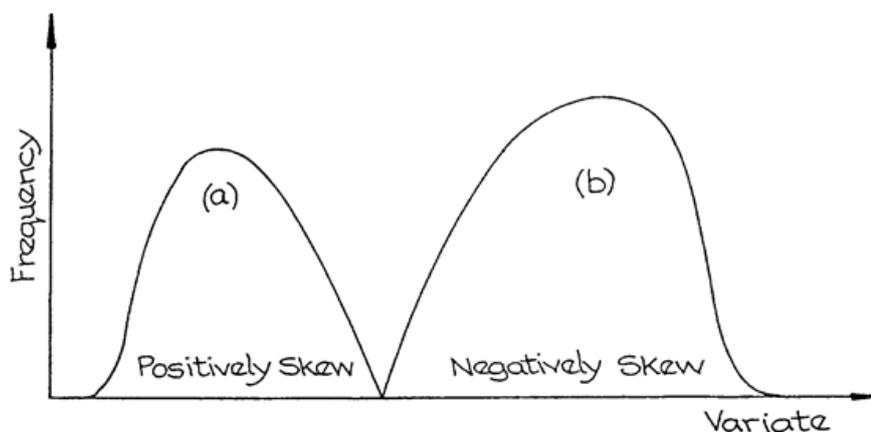


Figure 6.3

What, then, does skewness tell us? It tells us that we are to expect a few unusually high values in a positively skew distribution or a few unusually low values in a negatively skew distribution.

If a distribution is symmetrical, the mean, mode and median all occur at the same point, i.e. right in the middle. But in a skew distribution the mean and the median lie somewhere along the side of the "tail", although the mode is still at the point where the curve is highest. The more skewed the distribution, the greater the distance from the mode to the mean and the median, but these two are always in the same order; working outwards from the mode, the median comes first and then the mean - see Figure 6.4.

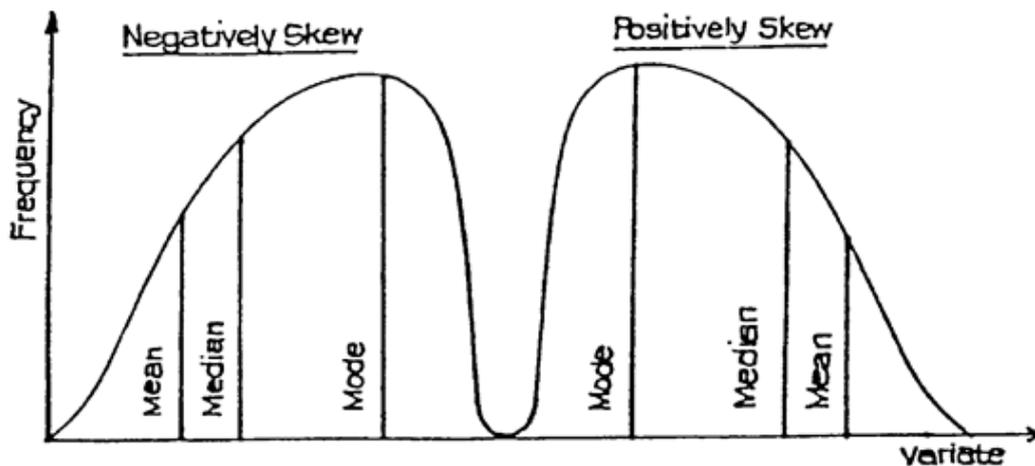


Figure 6.4

For most distributions, except for those with very long tails, the following relationship holds approximately:

$$\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$$

The more skew the distribution, the more spread out are these three measures of location, and so we can use the amount of this spread to measure the amount of skewness. The most usual way of doing this is to calculate:

$$\text{Pearson's First Coefficient of Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{SD}}$$

As we have seen, however, the mode is not always easy to find and so we use the equivalent formula:

$$\text{Pearson's Second Coefficient of Skewness} = \frac{3(\text{Mean} - \text{Median})}{\text{SD}}$$

You are expected to use one of these formulae when an examiner asks for the skewness (or "coefficient of skewness", as some of them call it) of a distribution. When you do the calculation, remember to get the correct sign (+ or -) when subtracting the mode or median from the mean and then you will get negative answers from negatively skew distributions, and positive answers for positively skew distributions. The value of the coefficient of skewness is between -3 and +3, although values below -1 and above +1 are rare and indicate very skewed distributions.

Examples of variates with positive skew distributions include size of incomes of a large group of workers, size of households, length of service in an organisation, and age of a workforce. Negative skew distributions occur less frequently. One such example is the age at death for the adult population in Rwanda.

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G. AVERAGES AND MEASURES OF DISPERSION

Measures of Central Tendency and Dispersion

- Averages and variations for ungrouped and grouped data.
- Special cases such as the Harmonic mean and the geometric mean

In the last section we described data using graphs, histograms and Ogives mainly for grouped numerical data. Sometimes we do not want a graph; we want one figure to describe the data.

One such figure is called the average. There are three different averages, all summarise the data with just one figure but each one has a different interpretation.

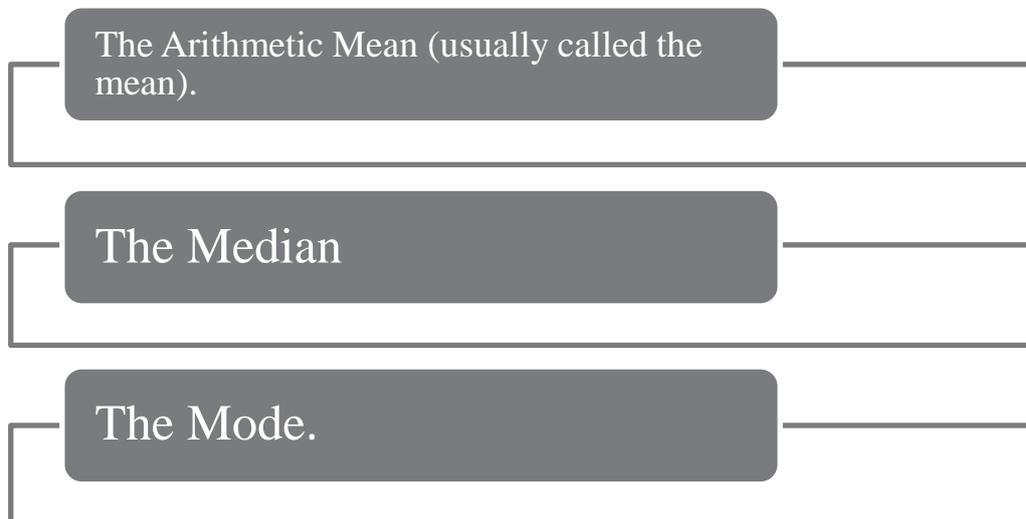


Figure 6.5

When describing data the most obvious way and the most common way is to get an average figure. If I said the average amount of alcohol consumed by Rwandan women is 2.6 units per week then how useful is this information? Usually averages on their own are not much use; you also need a measure of how spread out the data is. We will deal with the spread of the data later.

If you take the following 11 results. Each of the figures represents a student's results.

$x = 10, 55, 65, 30, 89, 5, 87, 60, 55, 37, 35.$

What is the average mark?

$$\text{Mean} = \frac{\sum x}{n} = \frac{528}{11} = 48$$

Median = the figure which half the class got below. To get the median, the data must be ranked.

Median = 5, 10, 30, 35, 37, **55**, 55, 60, 65, 87, 89

Median

Mode = the most frequently occurring result = 55.

Question

A random sample of 5 weeks showed that a cruise agency received the following number of weekly specials to the Caribbean:

20 73 75 80 82

- (a) Compute the mean, median and mode
- (b) Which measure of central tendency best describes the data?

From the above example concerning student's results, the mean figure is less than the median figure so if you wished to give the impression to your boss that the results were good you would use the median as the average rather than the mean. In business therefore when quoted an average number you need to be aware which one is being used.

The range = largest number – smallest number = 89 – 5 = 84 gives an idea of how spread out the data is. This is a useful figure if trying to analyse what the mean is saying. In this case it would show that the spread of results was very wide and that perhaps it might be better to divide the class or put on extra classes in future. Remember that the statistics only give you the information; it is up to you to interpret them. Usually in order to interpret them correctly you need to delve into the data more and maybe do some further qualitative research.

What is the best average, if any, to use in each of the following situations? Justify each of your answers.

(a) To establish a typical wage to be used by an employer in wage negotiations for a small company of 300 employees, a few of whom are very highly paid specialists.

(b) To determine the height to construct a bridge (not a draw bridge) where the distribution of the heights of all ships which would pass under is known and is skewed to the right.

There are THREE different measures of AVERAGE, and three different measures of dispersion. Once you know the mean and the standard deviation you can tell much more about the data than if you have the average only.

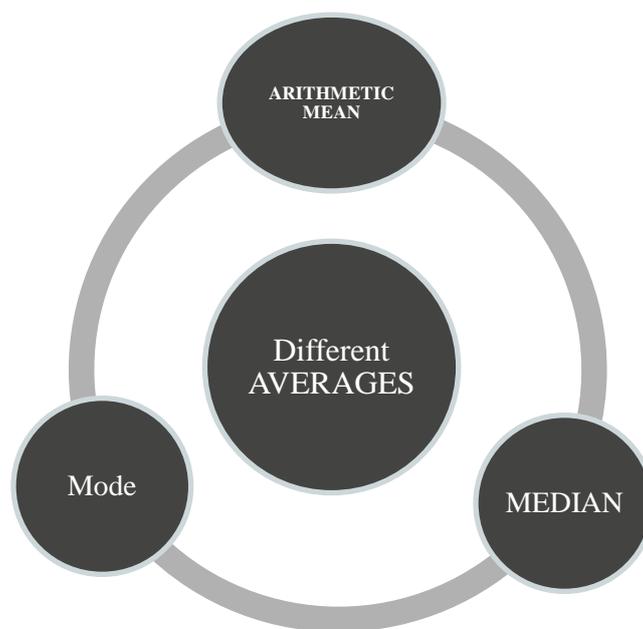


Figure 6.6

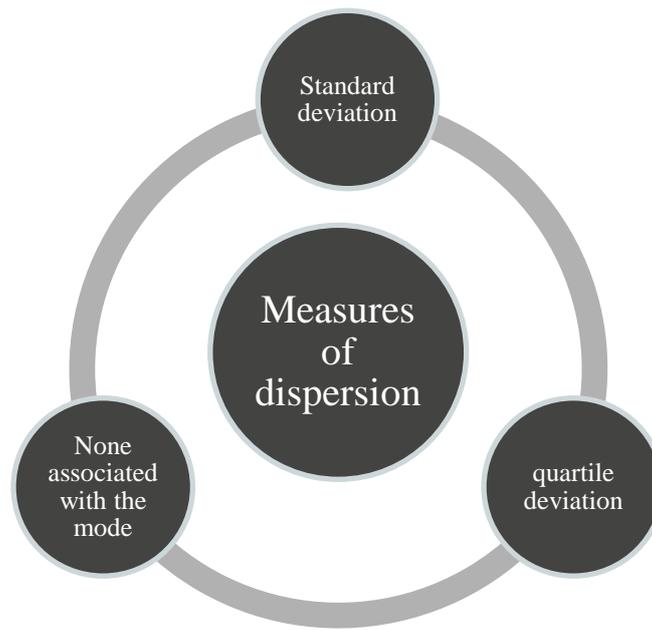


Figure 6.7

The Mean and Standard Deviation.

This is very important.

The mean of grouped data is more complex than for raw data because you do not have the raw figures in front of you, they have already been grouped. To find the mean therefore you need to find the midpoint of each group and then apply the following formula:

$$\text{Mean} = \frac{\sum fx}{\sum f} \text{ where } x \text{ represents the midpoint of each class and } f \text{ represents the frequency}$$

of that class. Note that if you are given an open ended class then you must decide yourself what that the mid- point is. The midpoint between $5 < 10 = 7.5$. The midpoint of a class < 10 you could say is 5 or 8 or whatever you want below 10, it depends on what you decide is the lower bound of the class.

If you need to get the midpoint of a class $36 < 56$ the easiest way is to add $36 + 56$ and divide by $2 = 46$.

Like all maths you just need to understand one example and then all the others follow the same pattern. You do need to understand what you are doing though because in your exam

you may get a question which has a slight trick and you need to be confident enough to figure out the approach necessary to continue.

Using the example we had in the last section on statistics grades, we will now work out the average grade.

Results	f	<i>X Mid point</i>	fx
0 but less than 20	8	10	80
20 but less than 30	9	25	225
30 but less than 40	11	35	385
40 but less than 50	14	45	630
50 but less than 60	11	55	605
60 but less than 70	10	65	650
70 but less than 80	9	75	675
80 but less than 90	6	85	510
90 but less than 100	2	95	190
<u>Total</u>	<u>80</u>	-	<u>3950</u>

Table 6.6

The mean score from the grouped data is given by the letter $\mu = \frac{\sum fx}{\sum f} = \frac{3950}{80} = 49.38$

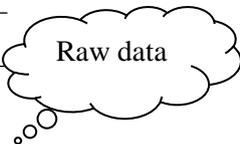
The Do-It-Better Manufacturing Company operates a shift loading system whereby 60 employees work a range of hours depending on company demands. The following data was collected:

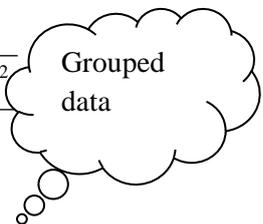
Hours worked	No. of employees
16 < 20	1
20 < 24	2
24 < 28	3
28 < 32	11
32 < 36	14
36 < 40	12
40 < 44	9
44 < 48	5
48 < 52	3

Table 6.7

The Standard Deviation

The next thing to estimate is the standard deviation. This is one figure which gives an indication of how spread out the data is. In the above example the number of hours worked is between 16 and 52 which is not that spread out so the standard deviate should be about 7(a rule of thumb is that 3 standard deviations should bring you from the mean to the highest or lowest figure in the data set). The mean here is 36, so if we take $36-16 = 20$ and divide by 3 we get 7 approx., or we could take $52-36 = 16 / 3 = 5.3$. So we take the bigger figure. However this is just a simple estimate and not sufficient for your exam. For your exam you need to apply the formula so you need to be able to work through it.

$$S.D = \sqrt{\frac{\sum (X - \bar{X})^2}{n}}$$


$$S.D = \sqrt{\frac{\sum f(X - \bar{X})^2}{\sum f}}$$


We will work through an example for finding the standard deviation for raw data first:

Find the standard deviation of the following 5 numbers:

X= 10, 20, 30, 40, 50.

The mean is 30.

Using the table below: The standard deviation equals: $\sqrt{\frac{1000}{5}} = \sqrt{200} = 14.14$

Mid pt	Mean	Deviations	
X	\bar{X}	$X - \bar{X}$	$(X - \bar{X})^2$
10	30	-20	400
20	30	-10	100
30	30	0	0
40	30	10	100
50	30	20	400
			1000

To work out the standard deviation for the grouped data using the example of the statistics score we use the formula for the grouped data which is nearly the same as for the raw data except you need to take into account the frequency with which each group score occurs.

To work out the standard deviation you continue using the same table as before. Look at the headings on each column. It follows the formula. You need to practice this.

Results	f	<i>X</i> Mid point	fx	Mean \bar{x}	$X - \bar{X}$	$(X - \bar{X})^2$	$f(X - \bar{X})^2$
0 but less than 20	8	10	80	49.38	-39.38	1550.78	12406.28
20 but less than 30	9	25	225	49.38	-24.38	594.38	5349.46
30 but less than 40	11	35	385	49.38	-14.38	206.78	2274.63
40 but less than 50	14	45	630	49.38	-4.38	19.18	268.58
50 but less than 60	11	55	605	49.38	5.62	31.58	347.43
60 but less than 70	10	65	650	49.38	15.62	243.98	2439.84
70 but less than 80	9	75	675	49.38	25.62	656.38	5907.46
80 but less than 90	6	85	510	49.38	35.62	1268.78	7612.71
90 but less than 100	2	95	190	49.38	45.62	2081.18	4162.37
Total	80	-	3950	-	-	-	40768.75

Table 6.8

So the standard deviation for the statistics scores is: $\sqrt{\frac{40768.75}{80}} = 22.57$

The Median and the Quartiles.

The median is the figure where half the values of the data set lie below this figure & half above. In a class of students the median age would be the age of the person where half the class is younger than this person and half older. It is the age of the middle aged student.

If you had a class of 11 students, to find the median age, you would line up all the students starting with the youngest to the oldest. You would then count up to the middle person, the 5th one along, ask them their age and that is the median age.

△ △ △ △ △ **△** △ △ △ △ △

To find the median of raw data you need to firstly rank the figures from smallest to highest and then choose the middle figure.

For grouped data it is not as easy to rank the data because you don't have single figures you have groups. There is a formula which can be used or the median can be found from the ogive. From the ogive, you go to the half way point on the vertical axis (if this is already in percentages then up to 50%) and then read the median off the horizontal axis.

If we use the data from the example of the statistics results we used before, you will remember we drew the ogive from the following data:

Table 6.9

Less than	Cumulative frequency	Percentage cumulative frequency.
20	8	10
30	17	21.25
40	28	35
50	42	52.5
60	53	66.25
70	63	78.75
80	72	90
90	78	97.5
100	80	100

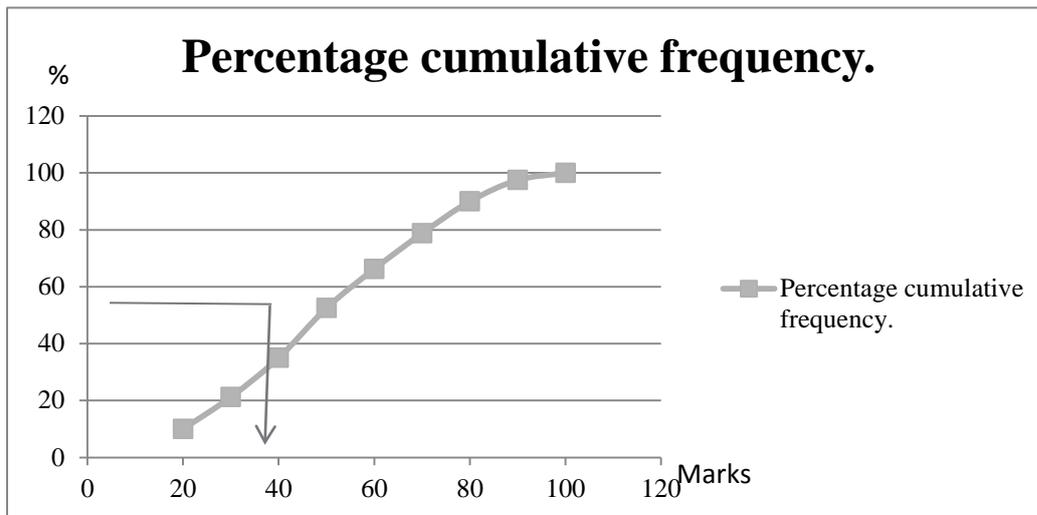


Figure 6.8

We can read the median off this and we can also read the quartiles. The median is read by going up to 50% on the vertical axis and the reading the mark off the horizontal axis. In the above example it is approximately 48 marks.

Using the formula we can also get the median: the formula is:

$$\text{Median} = L_m + \left[\frac{\frac{N}{2} - F_{m-1}}{f_m} \right] c_m$$

To use the formula you take the data in its frequency distribution as follows

Table 6.10

Results	Frequency
0 but less than 20	8
20 but less than 30	9
30 but less than 40	11 (Cumulative 28)
40 but less than 50	14
50 but less than 60	11
60 but less than 70	10
70 but less than 80	9
80 but less than 90	6
90 but less than 100	2
Total	80

$$\begin{aligned}
 \text{Median} &= 40 + \left[\frac{40 - 28}{14} \right] 10 \\
 &= 40 + \left(\frac{12}{14} \right) 10 \\
 &= 40 + 8.57 \\
 &= 48.57
 \end{aligned}$$

The quartiles can be found also from the ogive or using a similar formula to that above.

Quartile 1 measures the mark below which 25% of the class got (33) and quartile 3 represents the mark below which 75% of the class got (68). These can be read off the ogive at the 25% mark and the 75% mark.

The interquartile range is found using the formula $Q_3 - Q_1$. This indicates the spread about the median. The semi-interquartile range (which is similar to the standard deviation) is the interquartile range divided by 2.

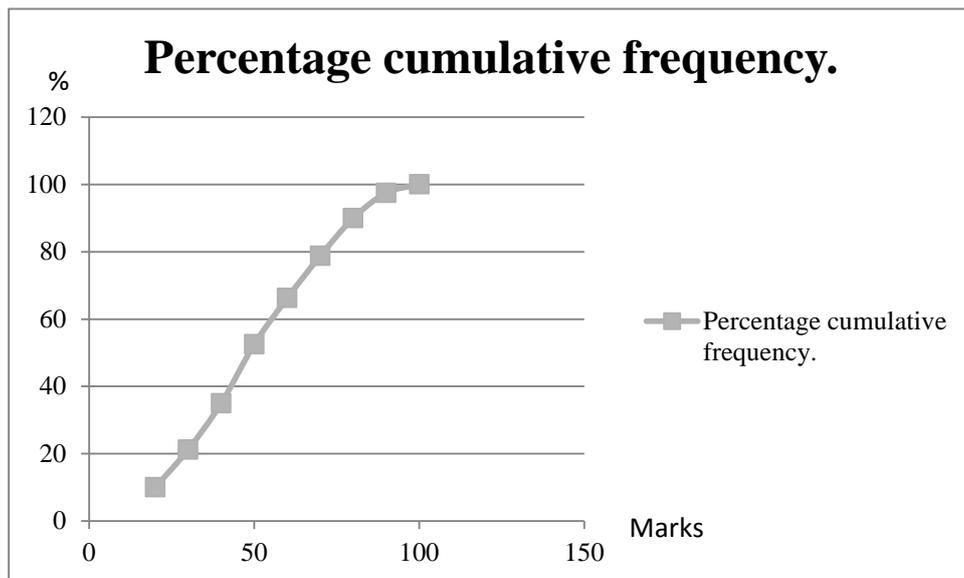


Figure 6.9

For data which is normally distributed the median should lie half way between the two quartiles, if the data is skewed to the right then the median will be closed to quartile 1. Why?

Percentiles are found in the same way as quartiles, the 10% percentile would be found by going up 10% of the vertical axis, etc.

The Mode

There is no measure of dispersion associated with the mode.

The mode is the most frequently occurring figure in a data set. There is often no mode particularly with continuous data or there could be a few modes. For raw data you find the mode by looking at the data as before, or by doing a tally.

For grouped data you can estimate the mode from a histogram by finding the class with the highest frequency and then estimating.

Formula for the mode:

$$\text{Mode} = L + \left[\frac{D_1}{D_1 + D_2} \right] \cdot C$$

To calculate the mode:

- 1) Determine the modal class, the class with the highest frequency
- 2) Find D_1 = difference between the largest frequency and the frequency immediately preceding it.
- 3) Find D_2 = difference between the largest frequency and the frequency immediately following it.

C= modal class width.

- **Measures of dispersion- range, variance, standard deviation, co-efficient of variation.**

The range is explained earlier it is found crudely by taking the highest figure in the data set and subtracting the lowest figure.

The variance is very similar to the standard deviation and measures the spread of the data. If I had two different classes and the mean result in both classes was the same, but the variance was higher in class B then results in class B were more spread out. The variance is found by getting the standard deviation and squaring it.

The standard deviation is done already.

The **co-efficient of variation** is used to establish which of two sets of data is relatively more variable.

For example, take two companies ABC and CBA. You are given the following information about their share price and the standard deviation of share price over the past year.

Table 6.11

	Mean	Standard deviation	Co efficient of variation(CV)
ABC	1.2	.8	.67
CBA	1.6	.9	.56

$$CV = \frac{\text{Standard Deviation}}{\text{Mean}} \text{ So CBA shares are relatively less variable.}$$

The Harmonic mean: The harmonic mean is used in particular circumstances namely when data consists of a set of rates such as prices, speed or productivity.

The formula for this is:

$$\text{Harmonic mean is: } \frac{n}{\sum \frac{1}{x}}$$

The Geometric mean: This is used to average proportional increases.

An example will illustrate the use of this and the application of the formula:

It is known that the price of a product has increased by 5% 2% 11% and 15% in four successive years.

The GM is:

$$\begin{aligned} & \sqrt[4]{1.05 \times 1.02 \times 1.11 \times 1.15} \\ & \sqrt[4]{1.367} \\ & = 1.081 \end{aligned}$$

Dispersion and Skewness:

The normal distribution is used frequently in statistics. It is not skewed and the mean, median and the mode will all have the same value. So for normally distributed data it does not matter which measure of average you use as they are all the same.

Shape of the normal distribution

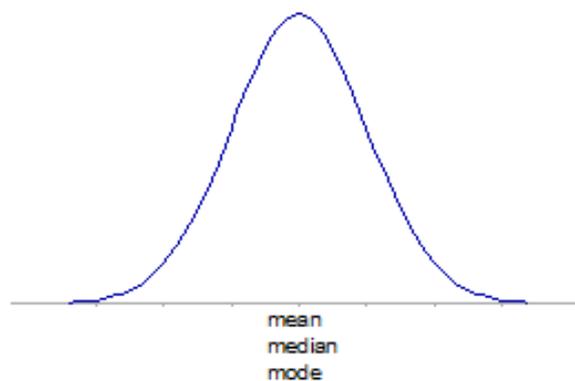


Figure 6.10

Data which is skewed looks like this:

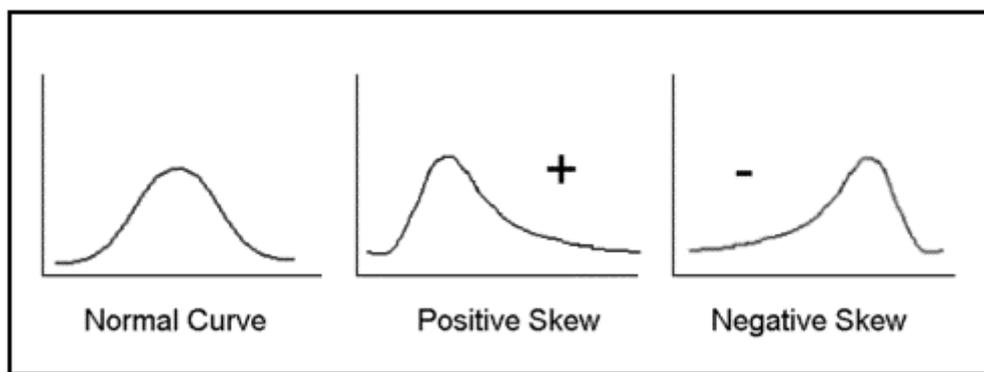


Figure 6.11

STUDY UNIT 7

The Normal Distribution; Statistical Inference

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A. INTRODUCTION

In Study Unit 4, Section E of this module, we considered various graphical ways of representing a frequency distribution. We considered a frequency dot diagram, a bar chart, a polygon and a frequency histogram. For a typical histogram, see Figure 7.1. You will immediately get the impression from this diagram that the values in the centre are much more likely to occur than those at either extreme.

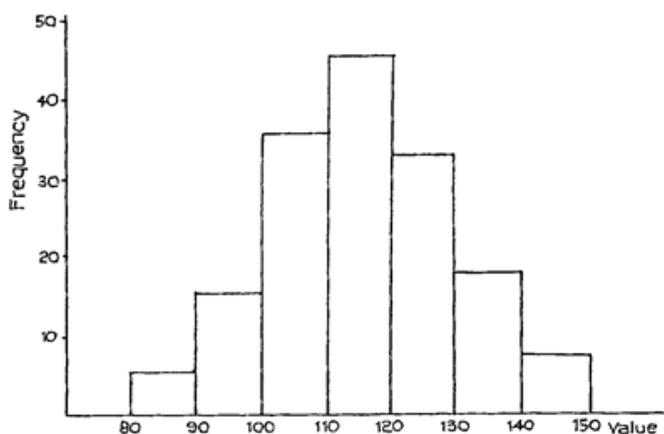


Figure 7.1

Consider now a continuous variable in which you have been able to make a very large number of observations. You could compile a frequency distribution and then draw a frequency bar chart with a very large number of bars, or a histogram with a very large number of narrow groups. Your diagrams might look something like those in Figure 7.2.

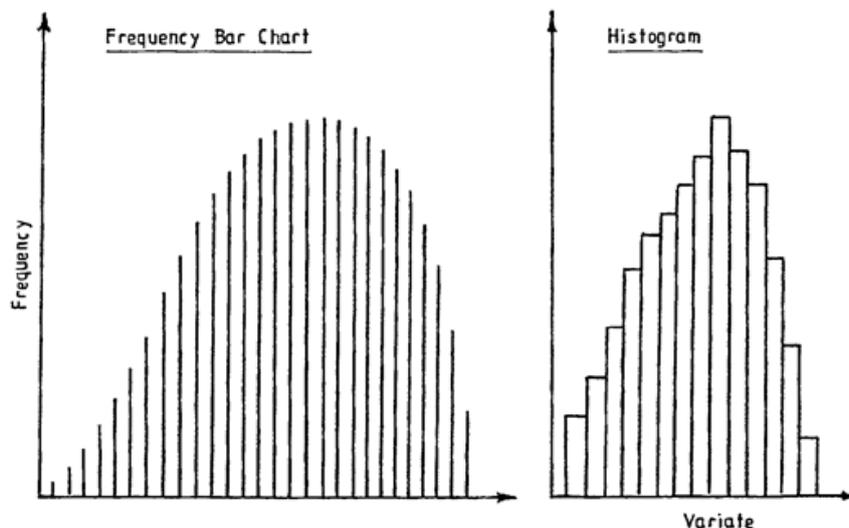


Figure 7.2

If you now imagine that these diagrams **relate to relative frequency distribution** and that a smooth curve is drawn through the tops of the bars or rectangles, you will arrive at the idea of a **frequency curve**.

Most of the distributions which we get in practice can be thought of as approximations to distributions which we would get if we could go on and get an infinite total frequency; similarly, frequency bar charts and histograms are approximations to the frequency curves which we would get if we had a sufficiently large total frequency. In this course, from now onwards, when we wish to illustrate frequency distributions without giving actual figures, we will do so by drawing the frequency curve, as in Figure 7.3.

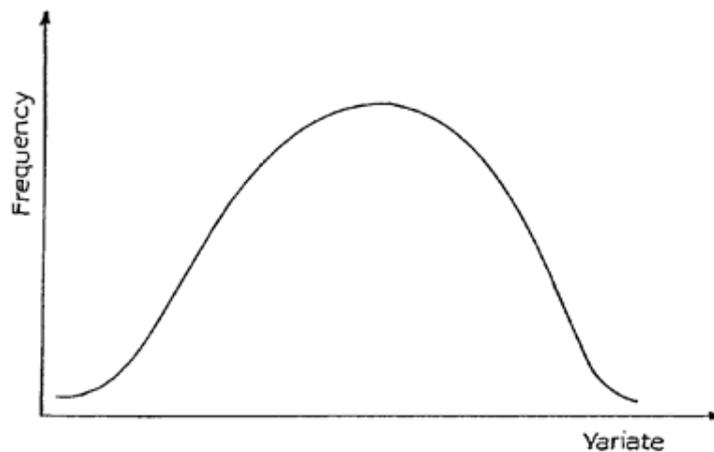


Figure 7.3

B. THE NORMAL DISTRIBUTION

The "Normal" or "Gaussian" distribution is probably the most important distribution in the whole of statistical theory. It was discovered in the early 18th century, because it seemed to represent accurately the random variation shown by natural phenomena. For example:

- heights of adult men from one race
- weights of a species of animals
- the distribution of IQ levels in children of a certain age
- weights of items packaged by a particular packing machine
- life expectancy of light bulbs

A typical shape is shown in Figure 7.4. You will see that it has a central peak (i.e. it is unimodal) and that it is symmetrical about this centre.

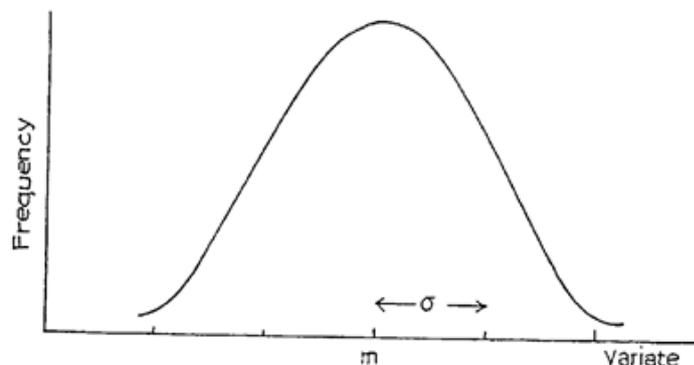


Figure 7.4

The mean of this distribution is shown as m on the diagram, and is located at the centre. The standard deviation, which is usually denoted by s , is also shown.

There are some interesting properties which these curves exhibit, which allow us to carry out calculations on them. For distributions of this approximate shape, we find that 68% of the observations are within ± 1 standard deviation of the mean, and 95% are within ± 2 standard deviations of the mean. For the normal distribution, these figures are exact. See Figure 7.5.

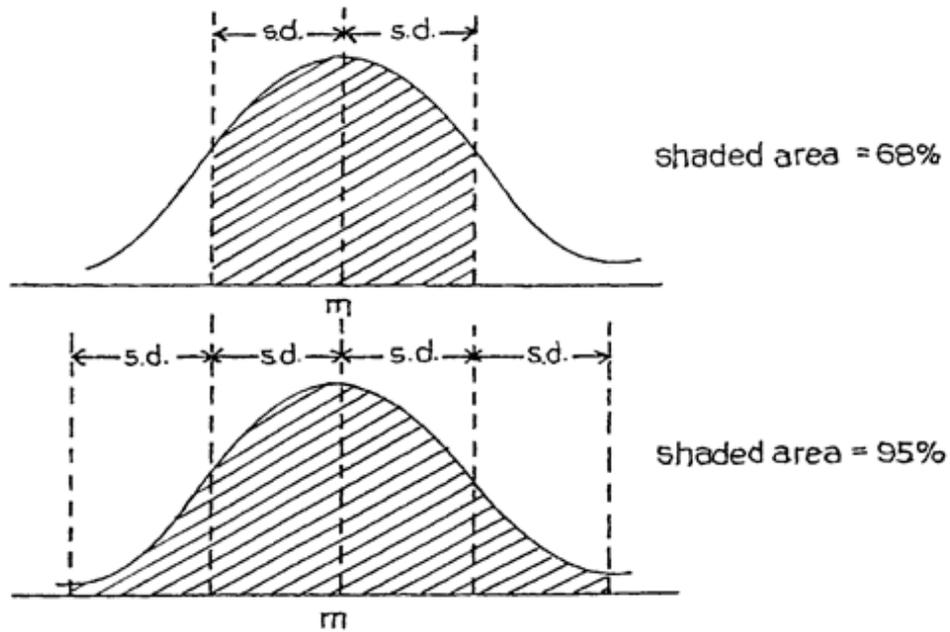


Figure 7.5

These figures can be expressed as probabilities. For example, if an observation x comes from a normal distribution with mean m and standard deviation s , the probability that x is between $(m - s)$ and $(m + s)$ is:

$$P(m - \sigma < x < m + \sigma) = 0.68$$

Also $P(m - 2\sigma < x < m + 2\sigma) = 0.95$

C. CALCULATIONS USING TABLES OF THE NORMAL DISTRIBUTION

Tables of the Normal Distribution

Tables exist which allow you to calculate the probability of an observation being within any range, not just $(m - s)$ to $(m + s)$ and $(m - 2s)$ to $(m + 2s)$. We show here a set of tables giving the proportion of the area under various parts of the curve of a normal distribution.

Table 7.1

AREAS IN TAIL OF THE NORMAL DISTRIBUTION

$\frac{(x - \mu)}{\sigma}$.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0916	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
3.0	.00135									

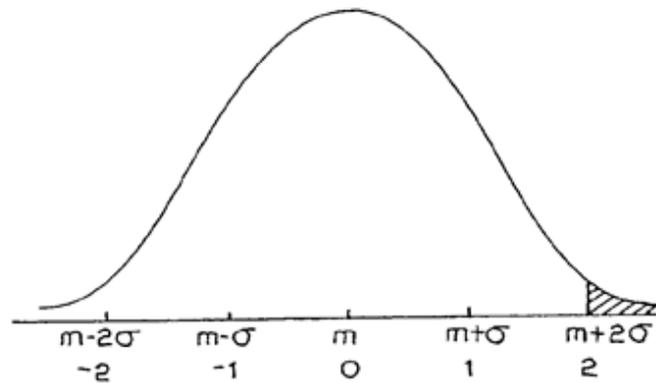


Figure 7.6

The figure given in the tables is the proportion of the area in **one tail** of the distribution. The area under a section of the curve represents the proportion of observations of that size. For example, the shaded area shown in Figure 48 represents the chance of an observation being greater than $m + 2s$. The vertical line which defines this area is at $m + 2s$. Looking up the value 2 in the table gives:

$$P(x > m + 2\sigma) = 0.02275$$

which is just over 2%.

Similarly, $P(x > m + 1\sigma)$ is found by looking up the value 1 in the tables. This gives:

$$P(x > m + 1\sigma) = 0.1587$$

which is nearly 16%.

You can extract any value from $P(x > m)$ to $P(x > m + 3\sigma)$ from the tables. This means that you can find the area in the tail of the normal distribution wherever the vertical line is drawn on the diagram.

Using the Symmetry of the Normal Distribution

Negative distances from the mean are not shown in the tables. Since the distribution is symmetrical, it is easy to calculate these.

$$P(x < m - 5\sigma) = P(x > m + 5\sigma)$$

$$\text{So } P(x > m - 5\sigma) = 1 - P(x < m - 5\sigma)$$

This is illustrated in Figure 7.7.

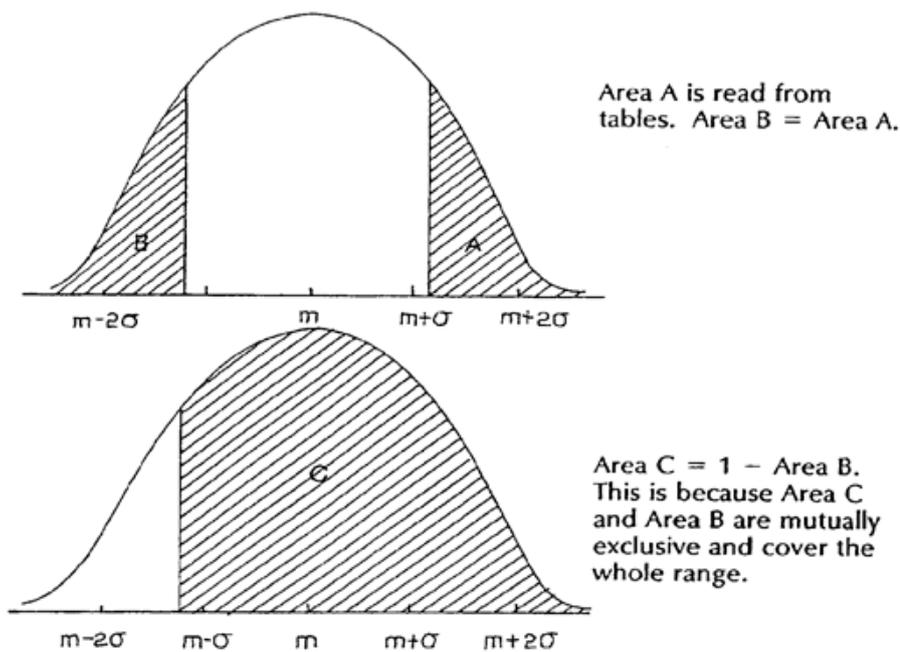


Figure 7.7

Further Probability Calculations

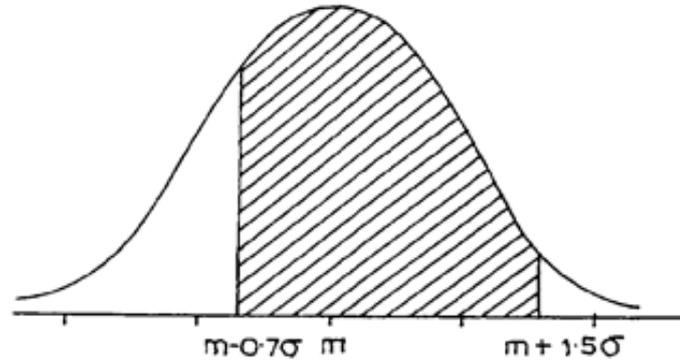


Figure 7.8

It is possible to calculate the probability of an observation being in the shaded area shown in Figure 7.8, using values from the tables. This represents the probability that x is between $m - 0.7\sigma$ and $m + 1.5\sigma$

$$\text{i.e. } P(m - 0.7\sigma < x < m + 1.5\sigma).$$

First find $P(x > m + 1.5\sigma) = 0.0668$ from the tables.

Then find $P(x < m - 0.7\sigma)$ or $P(x > m + 0.7\sigma)$

$$= 0.2420 \text{ since the distribution is symmetrical.}$$

The proportion of the area under the curve which is shaded in Figure 7.8 is $1 - 0.0668 - 0.2420$ or 0.6912 .

Hence $P(m - 0.7\sigma < x < m + 1.5\sigma) = 0.6912$.

Example

A production line produces items with a mean weight of 70 grams and a standard deviation of 2 grams. Assuming that the items come from a normal distribution, find the probability that an item will weigh 65 grams or less.

65 grams is 5 grams below the mean.

Since the standard deviation is 2 grams, this is 2.5 standard deviations below the mean.

Let x be the weight of an individual item.

$$\begin{aligned}P(x < 65) &= P(x < m - 2.5\sigma) \\ &= P(x > m + 2.5\sigma) \\ &= 0.00621 \text{ from the tables.}\end{aligned}$$

Now find the probability that an item will weigh between 69 and 72 grams.

69 grams is 0.5 standard deviations below the mean, and 72 grams is 1 standard deviation above the mean.

Therefore find $P(m - 0.5\sigma < x < m + \sigma)$.

$$P(x > m + \sigma) = 0.1587$$

$$P(x < m - 0.5\sigma) = P(x > m + 0.5\sigma) = 0.3085.$$

So, $P(m - 0.5\sigma < x < m + \sigma) = 1 - 0.1587 - 0.3085$

or $P(69 < x < 72) = 0.5328$

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D. STATISTICAL INFERENCE

Introduction

Often, in real life, the mean and standard deviation of a particular population of items are not known. The population may be too large to measure them all, or it may be continuous production, and therefore impossible to measure all items. For example, consider a machine which packages a food product. These packages are marked as 500 grams and are sold by weight. The average weight of the whole population of packages is clearly important, but is unknown. The only way to arrive at an estimate is to take a sample of items from the population. Having taken a sample, you can then calculate the sample mean. Is this a good indication of the population mean? How close is it likely to be? Statistical inference is the technique of using sample statistics to estimate population statistics.

Sampling Distributions

If a sample is taken from a population, and the sample mean calculated, this can be used to estimate the population mean. Now consider what happens if a second sample is taken from the same population. Another estimate is obtained, which will probably be different from the first one. Imagine now that a large number of samples are taken, all from the same population. You can see that all the estimates obtained for the population mean can themselves be used to give a distribution. That is, the sample means have a distribution, in just the same way that the population weights have a distribution. In fact, there are very important similarities as follows:

- a) If the population distribution is normal, the distribution of the sample means will also be normal.
- b) The mean of the distribution of the sample means equals the mean of the population distribution.
- c) The standard deviation of the distribution of sample means is smaller than the standard deviation of the population. In fact:

$$\text{S.d. of sample means} = \frac{\text{S.d. of population}}{\sqrt{n}}, \text{ where } n \text{ is the sample size.}$$

Figure 7.9 expresses, this graphically. The wider, lower distribution is the original population of weights, and the narrower, taller distribution is the distribution of the sample means. This is the curve you would obtain with samples of size 4.

$$\begin{aligned} \text{S.d. of sample means} &= \frac{\text{S.d. of population}}{\sqrt{4}} \\ &= \frac{\text{S.d. of population}}{2} \end{aligned}$$

The two distributions are centred around the same mean, and one has a standard deviation which is half that of the other. For larger sample sizes, the standard deviation of the sample means would be even smaller.

The standard deviation of the sample means is called the "standard error". The meaning is the same, but this term is used because it is for a **sample distribution** rather than the distribution for the original population.

These figures can be used to carry out statistical tests. In the next study unit, we will expand on these ideas.

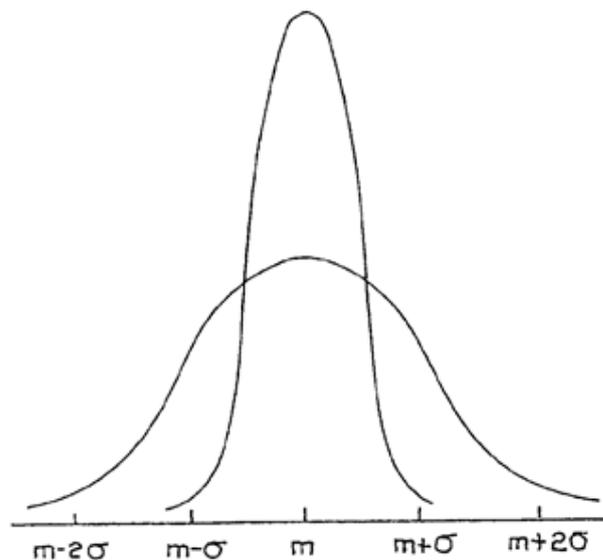


Figure 7.9

Samples Taken from Non-Normal Distributions

In some instances we may not know whether our population is truly normal, or we may know that it is not normal and still wish to estimate values for our population by taking samples. The result in the previous section still holds if the population size is reasonably large. It is difficult to say how large it must be, but the closer the distribution is to a normal distribution, the smaller the population size is required to be, for the result to hold, i.e. the sample means follow a normal distribution with

$$\begin{aligned} \text{mean of sample means} &= \text{population mean} \\ \text{s.d. of sample means} &= \frac{\text{S.d. of population}}{\sqrt{n}} \end{aligned}$$

where n is the sample size.

This important fact is called the "Central Limit Theorem". It allows us to estimate the population mean using the sample mean. We can also estimate the population standard deviation using the sample standard deviation.

Expressing this another way, we can use the sample mean as an estimate of the population mean, because the expected value (or mean) of the distribution of sample means is the same as the population mean, and the distribution of sample means is more closely clustered around this mean than the original distribution values.

Combining Normal Distributions

Consider now the situation where we have two distributions, which may be different. For example, a production line consists of two machines, both producing the same goods. A sample is taken from each and the following results are obtained:

Sample 1	Sample 2
mean = \bar{x}_1	mean = \bar{x}_2
s.d. = σ_1	s.d. = σ_2
sample size = n_1	sample size = n_2

We can estimate both the sum and the difference of the means, and also calculate a standard error for this estimate.

Sum of means:

$$\text{Estimate} = \bar{x}_1 + \bar{x}_2$$

$$\text{s.e. of estimate} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Difference of means:

$$\text{Estimate} = \bar{x}_1 - \bar{x}_2$$

$$\text{s.e. of estimate} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

STUDY UNIT 8

Estimation and Confidence Intervals; Statistical Testing

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A. INTRODUCTION

In the previous study unit we discussed the idea of taking a sample of items from a much larger population. Calculations may be carried out on the sample data, and the results used to make inferences regarding the whole population. We discussed the idea that with repeated samples, the statistic calculated from the sample would vary. In fact, the statistic will have a distribution in just the same way that the population of items has a distribution. Although it will not be the same distribution, where the sample statistic is the mean, there are known similarities.

In reality, we will probably only take one sample, and use the sample mean to estimate the population mean. However, because of the theory regarding the distribution of all sample means, we are able to say how close this estimate is likely to be. That is, we are able to put a confidence interval around our estimate.

Remember that all the results in this study unit are general ones for samples taken from large populations, and therefore because of the central limit theorem, the original population from which the sample is taken does not need to follow the normal distribution.

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B. ESTIMATION OF A POPULATION MEAN, WITH A CONFIDENCE INTERVAL, FROM SAMPLE DATA

Example Using a 90% Confidence Interval

The best way to illustrate the idea of a confidence interval is using an example. We wish to estimate the average amount of time spent by adults in watching TV in one week, for the town in which we live. In order to do this, we have taken a random sample of 20 people from the town, and asked them to keep records to the nearest quarter of an hour. The results look like this:

Table 8.1

20.5	32.0	19.5	18.25	21.5
25.75	15.25	8.0	17.0	21.0
23.25	26.25	20.0	14.0	18.0
18.75	16.75	23.0	22.25	20.5

$$\text{The mean } \bar{x} = \frac{\sum x}{n} = \frac{401.5}{20} = 20.075 \text{ hours}$$

In previous study units we have used the standard deviation formula which follows:

$$\text{S.d.} = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

However, when we are dealing with small samples taken from large populations, and we wish to use the sample standard deviation to estimate the population standard deviation, a slightly modified formula is preferred:

$$\text{S.d.} = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} \text{ for grouped distributions}$$

This formula is known to provide a better estimate. Using this formula

for our sample data we get a standard deviation of 5.002 hours.

Therefore, to summarise for our sample:

Mean, $\bar{x} = 20.075$

Standard deviation, $\sigma = 5.002$

Sample size, $n = 20$.

This implies that the population mean can be estimated as 20.075 hours. However, we wish to know how accurate this estimate is likely to be. We know that for large populations, the theoretical distribution which the sample means follow is normal, even if the population is not normal. We know also that it has the same mean as the population mean, and that it has a standard error given by:

$$\text{S.e.} = \frac{\text{Population s.d.}}{\sqrt{n}} \quad \text{where } n \text{ is the sample size.}$$

Hence in this example, the standard error of the mean is:

$$\text{S.e.} = \frac{5.002}{\sqrt{20}} = 1.118 \text{ hours}$$

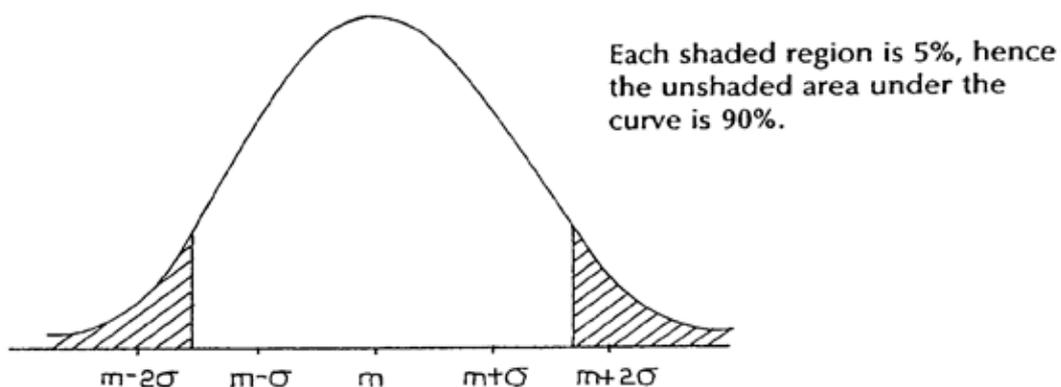


Figure 8.1

Figure 8.1 illustrates the fact that if we wish to find a 90% confidence interval, we must allow 5% in each tail of the distribution. Hence we must use the tables of the normal distribution in reverse, to look up 0.05, which is 5% expressed as a decimal. The nearest we can get to this is 0.0505, so reading off the value which corresponds to this, we get 1.64 standard deviations. We can be 90% confident that the true number of viewing hours per week is within our sample mean ± 1.64 standard deviations.

Hence 90% confidence interval for the average number of viewing hours per week

$$\begin{aligned}
 &= 20.075 \pm 1.64 \times 1.118 \\
 &= 20.075 \pm 1.834 \\
 &= 20.08 \pm 1.83 \text{ to 2 decimal places.}
 \end{aligned}$$

This region is defined as the 90% confidence interval. We are 90% confident that the population mean (number of viewing hours per week) is between 18.25 and 21.91 hours.

Other Sizes for the Confidence Interval

Table 8.2

Size of Confidence Interval	Area in each Tail	Value from Tables
95%	20 ½ % or 0.025	1.96
98%	1% or 0.01	2.33
99%	½ % or 0.005	2.58

Since the tables of the normal distribution cover the whole range up to ± 3 standard deviations from the mean, any size of confidence interval may be constructed. Table 8.2 shows some of the usual ones which are used. You should verify that you can obtain the values from the table, so that in an examination, you can read off any value as requested.

For our example, this gives the following results:

95% confidence interval -

$$20.075 \pm 1.96 \times 1.118 \text{ or } 17.89 \text{ to } 22.27 \text{ hours}$$

98% confidence interval -

$$20.073 \pm 2.33 \times 1.118 \text{ or } 17.47 \text{ to } 22.68 \text{ hours}$$

99% confidence interval -

$$20.075 \pm 2.58 \times 1.118 \text{ or } 17.2 \text{ to } 22.96 \text{ hours.}$$

You will notice that the larger the confidence we wish to place in our results, the wider the range of estimates becomes.

C. ESTIMATES OF A POPULATION PROPORTION, WITH A CONFIDENCE INTERVAL, USING SAMPLE DATA

Another estimate which is commonly required is the estimate of a proportion. For example:

- What proportion of all invoices contain errors?
- What proportion of the population use our product?

The sampling distribution of a proportion is known to follow these rules:

- a) The sampling distribution of proportions is a normal distribution.
- b) The mean of the sampling distribution is equal to the mean of the population.
- c) The standard error of the sampling distribution is given by:

$$\text{S.e.} = \sqrt{\frac{p(1-p)}{n}}$$

where p is the sample proportion and n is the sample size.

We can use these results, when we have taken a sample, to estimate a population proportion and set up a confidence interval for this proportion, just as we did for population means.

Example

A random sample of 100 invoices is taken from the complete set of invoices for one financial year, and inspected for errors of any kind. It is found that 20 of them have errors.

Estimate the proportion of all invoices with errors and give 95% and 99% confidence intervals for this estimate.

$$\text{Sample proportion, } p = \frac{20}{100} = 0.2$$

$$\text{Sample size, } n = 100$$

$$\text{Sample standard error} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{100}} = 0.04$$

95% confidence interval:

$$= 0.2 \pm 1.96 \times 0.04$$

$$= 0.02 \pm 0.078$$

$$= 0.122 \text{ to } 0.278$$

or 12.2% to 27.8%

99% confidence interval:

$$= 0.2 \pm 2.58 \times 0.04$$

$$= 0.2 \pm 0.103$$

$$= 0.097 \text{ to}$$

or 9.7% to 30.3%

Again, any size of confidence interval can be produced, but the greater the confidence, the larger the interval.

D. CALCULATING THE SAMPLE SIZE

In the previous examples, we have been starting with a known sample size and using this to calculate a confidence interval. It is also possible to work this calculation the other way round. That is, we can start with the required confidence interval, and use it to calculate the size of sample we must take.

Example 1

I wish to estimate the average number of viewing hours in my population, from a sample, and I wish to be 99% confident that my estimate is correct within \pm half an hour.

The 99% confidence interval is 2.58 times the standard error of the mean. Hence we need:

$$2.58 \frac{\sigma}{\sqrt{n}} = 0.5$$

$$\begin{aligned} \text{or } n &= \left(\frac{2.58\sigma}{0.5} \right)^2 \\ &= \left(\frac{2.58 \times 5.002}{0.5} \right)^2 \\ &= 666. \end{aligned}$$

Note that we need an estimate of s , the standard deviation in the population, and may therefore be required to carry out a small initial survey to estimate this.

Example 2

I wish to be 95% confident that my estimate of the proportion of invoices containing errors is within $\pm 3\%$. What size sample should I take?

$$3\%, \text{ or } 0.03 = 1.96 \frac{p(1-p)}{n}$$

$$\begin{aligned} \text{Hence } n &= \left(\frac{1.96}{0.03} \right)^2 p(1-p) \\ &= 683. \end{aligned}$$

Therefore I should take a sample of size 683. Note also that we need to know p to estimate the sample size. Again, we may be required to carry out a small initial survey to estimate this, if we have no idea of its expected value. However, if we have a rough idea of its size, that will probably be sufficiently accurate for a sample size calculation.

E. STATISTICAL TESTS

This topic is also called "hypothesis testing" or "significance testing". This is because the method used is to set up a hypothesis and then carry out a test of this hypothesis using sample data, at a given significance level.

The approach is straightforward. A hypothesis is proposed. Then a sample is taken. The likelihood of the sample result, given that the hypothesis is true, is calculated. If this is within the significance level which we wish to use, the hypothesis is accepted. If not, the hypothesis is rejected.

The original hypothesis is called the **null** hypothesis, and by convention is denoted by H_0 .

Example 1

A machine has been set up to fill jars with 250 grams of jam. A sample of 100 jars showed that the average weight of jam was 251 grams, with a standard deviation of 4 grams. Has the machine been wrongly adjusted?

We set up a null hypothesis that the mean is equal to 250 grams. We write this as:

$$H_0: \mu = 250 \quad \text{where } \mu \text{ denotes the population mean.}$$

The alternative to this, which will be accepted if the sample results do not conform to H_0 , is:

$$H_1: \mu \neq 250.$$

Example 2

Consider now a slightly different question. Have the jars been overfilled? Now we have:

$$H_0: m = 250$$

and

$$H_1: m \neq 250.$$

Tests are by convention carried out at either the 5% significance level, or the 1% significance level, although any level is possible. If our sample result has less than a 5% (or a 1%) chance of occurring by chance, given that H_0 is true, then we will reject H_0 .

These significance levels are analogous to the 95% and 99% confidence intervals which we discussed previously. If the confidence interval is 95%, this represents the area in the centre of the normal distribution curve, and the area outside this, in the tails, is 5%.

F. ONE- AND TWO-TAILED TESTS

Example 1

We have our null hypothesis:

$$H_0: \mu = 250$$

and our alternative hypothesis:

$$H_1: \mu \neq 250.$$

We will carry out a test at the 5% level of significance.

How closely do the sample results conform to H_0 ?

$$\text{Sample mean} = 251$$

$$\text{Standard error of mean} = \frac{4}{\sqrt{100}} = 0.4$$

$$\text{Sample size} = 100$$

Let us calculate how far the sample mean is from our assumed mean of 250. It is:

$$251 - 250 = 1 \text{ gram}$$

over, but we must express this in terms of our standard error.

$$\text{The sample statistic } Z = \frac{251 - 250}{0.4} = 2.5$$

That is, the sample mean is $2 \frac{1}{2}$ times the standard deviation above the value of 250. How likely is this to occur by chance? Is it greater or less than our 5% significance level? You will remember that the 5% level, with $2 \frac{1}{2}$ % in each tail, occurs at 1.96 standard deviations. (See Table 8.2, or use the tables of the normal distribution.) Our result is more extreme than this,

since $2.5 > 1.96$, so we reject H_0 and accept H_1 . The mean is not equal to 250 grams at the 5% significance level.

Now let us repeat this calculation at the 1% significance level. Again $Z = 2.5$, but how does this compare with the 1% level, or $\frac{1}{2}$ % in each tail? The 1% level, with $\frac{1}{2}$ % in each tail, is at 2.58 standard deviations from the mean. Now our value of Z is less extreme than this. Therefore at the 1% significance level, we accept $H_0; \mu = 250$. We have decided that there is not enough evidence to reject H_0 at this level of significance.

This example, at both significance levels, is an example of a two-tailed test. Our null hypothesis is that $\mu = 250$, but our alternative is that $\mu \neq 250$. We are allowing for a variation in either direction, and are not concerned with whether this is above or below the assumed mean. The practical effect of this is that when we use the normal tables, as with the confidence interval calculations, we must halve the significance to allow for both tails of the normal distribution.

Example 2

You will remember that for Example 1 we have:

$$H_0: \mu = 250$$

and

$$H_1: \mu > 250.$$

We are now looking only for overfill. We are not concerned with the underfill situation.

Again:

$$\text{Sample mean} = 251$$

$$\text{Standard error of mean} = \frac{4}{\sqrt{100}} = 0.4$$

$$\text{Sample size} = 100$$

and our sample statistic

$$Z = \frac{251 - 250}{0.4} = 2.5$$

1) 5% Significance

Since we are now concerned only with the possibility that there may be an overflow situation, our 5% can all be in the upper tail of the distribution. See Figure 8.2.

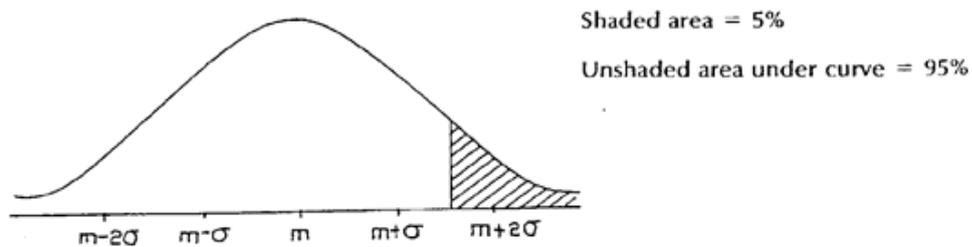


Figure 8.2

Hence, this time we look up 0.05 in the body of the normal distribution tables, to obtain a value of 1.64.

Our sample statistic, 2.5, is greater than the 1.64 standard error limit, so it lies in the shaded area. There is less than a 5% chance of obtaining such a value if H_0 is true. We therefore reject H_0 , on the basis of our sample, and accept $H_1: \mu > 125$.

2) 1% Significance

We look up 0.01 in the normal distribution tables, to obtain 2.37 as the test value.

Again $2.5 > 2.33$, so we reject H_0 .

With the one-tailed test, the null hypothesis is rejected at both 5% and 1% significance.

Example 3

Significance tests may also be carried out on proportions.

A local group claims at a public meeting that the majority of people living in the local area are against the proposed route for a new road. You decide to try to verify this. In a survey consisting of 165 people, 60 are found to have an objection. Test the group's claim at the 1% level of significance.

$H_0: \pi = 0.5$, i.e. the null hypothesis is that the proportion of people who are against the proposed route is 50%.

$H_1: \pi < 0.5$, i.e. we are carrying out a one-tailed test.

$$\begin{aligned}\text{Standard error} &= \sqrt{\frac{p(1-p)}{n}} \\ &= \sqrt{\frac{0.5(1-0.5)}{165}} \\ &= 0.039.\end{aligned}$$

$$\text{Sample proportion} = \frac{60}{165} = 0.364.$$

$$\text{Test statistic } Z = \frac{0.5 - 0.364}{0.039} = 3.49$$

That is, our sample statistic lies in the left-hand tail of the distribution, at a distance of 3.49 standard deviations from the mean. The 1% significance level, for a one-tailed test, is 2.37. Our test statistic, $3.49 > 2.37$, hence our result is more extreme and we reject the hypothesis that 50% of the people in the local area oppose the route for the new road. We accept the alternative that the number of people who oppose it is less than 50%.

G. STATISTICAL PACKAGES

There are many statistical packages, both for personal computers and for larger machines, which will carry out such tests. However, it is very important for you to have an understanding of the principles behind the tests. Computers carry out calculations blindly, and will always come up with an answer, even if the results are meaningless. You must know enough about the techniques to examine the results critically, and verify that all statistics and conclusions are reasonable.

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H. ANALYSIS AND INTERPRETATION OF SAMPLE DATA

Notes on sampling theory and the Chi-squared distribution.

This topic can be divided into 3 parts.

- 1) Confidence intervals.
- 2) Hypothesis testing.
- 3) The Chi-Squared test.

Confidence intervals

The formula for the confidence interval is:

$$\mu = \bar{X} \pm Z \cdot STEM$$

STEM = standard error of the mean and is found by the formula $\frac{\text{standard deviation}}{\sqrt{n}}$

Example:

A sample of **100** doctors is taken from the list of all doctors working in Rwanda. The mean salary of this sample of doctors is **RWF80000** per year with a standard deviation of **RWF3000**. Find a **95%** confidence interval for the mean salary of all doctors working in Rwanda.

100 = n (The number in our sample)

RWF80000 = \bar{X} (the mean of the sample)

RWF3000 = Standard deviation.

95% gives us the Z in the formula, which we know to be 1.96. If you were asked to get a 99% confidence interval Z = 2.58.

We can now slot these figures into the formula.

$$\mu = RWF80000 \pm 1.96 \cdot \frac{3000}{\sqrt{100}}$$

$$\mu = RWF80000 \pm 1.96 \cdot 300$$

$$\mu = RWF80000 \pm 588$$

This reads as follows: we are 95% sure that the mean salary of all doctors lies somewhere in the interval $RWF80000 \pm 588$.

The interval is (RWF79412, RWF80588)

Confidence intervals for proportions/percentages.

Often when doing a poll perhaps for an election or some marketing project we get 95% intervals for where the true percentage lies.

Example:

In a poll of 1000 college students 600 smoke. Find the 95% confidence interval for the percentage of all students who smoke.

The confidence interval is given by the equation:

$$\pi = p \pm Z_{step}$$
$$step = \sqrt{\frac{p(1-P)}{n}} \text{ or } \sqrt{\frac{p(100-P)}{n}}$$

In the above example $p = 600/1000 = 0.6$

$$\pi = 0.6 \pm 1.96 \sqrt{\frac{0.6(1-0.6)}{1000}}$$

$$\pi = 0.6 \pm 1.96(0.01549)$$

$$\pi = 0.6 \pm 0.03$$

This is interpreted as:

- The proportions of all students who smoke lies between 60% \pm 3%
- The interval is (57%, 63%)
- The 0.03 or 3% is called the margin of error.

Hypothesis testing

When carrying out a hypothesis test you need to follow the following 3 steps:

- 1) State hypothesis
- 2) Carry out test (Z test)
- 3) Draw conclusion.

There are 4 different types of hypothesis test:

<p>Hypothesis test for a single mean</p> <p>$H_0: \mu = \text{population mean}$</p> $Z = \frac{\bar{X} - \mu}{STEM}$	<p>Hypothesis test for a single proportion</p> <p>$H_0: \pi = \text{population proportion}$</p> $Z = \frac{p - \pi}{STEP}$
<p>Hypothesis test for differences in 2 sample means</p> <p>$H_0: \mu_a = \mu_b$</p> $Z = \frac{\mu_a - \mu_b}{STEDM}$	<p>Hypothesis test for differences in 2 sample proportions</p> <p>$H_0: \pi_a = \pi_b$</p> $Z = \frac{\pi_a - \pi_b}{STEDP}$

In all cases, if the absolute value of Z is less than 1.96 you accept H_0 .

The Chi-Squared test

This test is used to compare observed and expected values.

The test statistic is found using the formula

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

There are two main occasions when this is used.

The simple chi-squared test:

A dice is thrown 120 times with the following observed results:

Table 8.3

Face	Observed values
1	18
2	22
3	20
4	21
5	19
6	20

Do these results suggest the dice is fair?

To carry out this test we always assume before we start that the dice is fair. If it were fair, how many of each face would you expect to observe?

The answer is found by dividing 120 by 6, which equals 20.

Therefore, the expected values are 20.

Table 8.4

Face	Observed values	Expected values	X ²
1	18	20	0.2
2	22	20	0.2
3	20	20	0
4	21	20	0.05
5	19	20	0.05
6	20	20	0
Total	120	120	0.5

$$\frac{(18 - 20)^2}{20}$$

Testing at the 0.05 level with 5 degrees of freedom the critical value is 11.07. Since 0.5 is less than 11.07, we accept that the dice is fair.

In the simple chi-squared test the degrees of freedom is found by taking n-1.

The following are the results of a survey:

Table 8.5

Type of employee	Number of days off sick		
	Less than 5 days	5-10 days	More than 10 days
Monthly Paid	95	47	18
Weekly Paid	143	146	112

H_0 = There is no association between the number of days off sick and the type of employee.

To test this hypothesis you assume it is correct and based on that assumption you work out the expected values. The first thing you do is get the totals of each row and each column as done below:

Table 8.6

Type of employment	Number of days off sick			Total
	Less than 5 days	5-10 days	More than 10 days	
Monthly paid	95	47	18	160
Weekly paid	143	146	112	401
Total	238	193	130	561

The expected values are found by using the following formula:

$$\frac{\text{Total of the row} \times \text{total of the column}}{\text{grand total}}$$

The expected value associated with 95 in the first box is thus equal to:

$$\frac{(160)(238)}{561} = 67.9.$$

An expected value is thus found for each of the six observed values.

The chi-squared statistic is then calculated as follows:

Table 8.7

Observed values	Expected values	$(O-E)^2 / E$
95	67.9	10.816
47	55.0	1.164
18	37.1	9.833
143	170.1	4.318
146	138.0	.464
112	92.9	3.927
		Total = 30.522

The total is the chi-squared statistic.

STUDY UNIT 9

Index Numbers

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A. THE BASIC IDEA

Table 9.1 shows the monthly profits of Firm X for a period of one year. We could plot profits against time (i.e. each month) and draw a graph. However, if we are interested in **changes** in profits rather than in the actual level of profits, we can use one month's figures, say January, as standard and express all the others as percentages of this standard.

Because we are dealing with percentages, we use a standard figure of 100.

In Table 9.1, the right-hand column shows January set to the standard figure of 100 and all the other profit values set to percentages of this standard.

Table 9.1: Monthly Profits of Firm X

Month	Profit	Profit Based on Jan. = 100
Jan.	512	100
Feb.	520	102
Mar.	530	104
Apr.	531	104
May	546	107
Jun.	549	107
Jul.	560	109
Aug.	565	110
Sep.	568	111
Oct.	573	112
Nov.	584	114
Dec.	585	114

The percentage figures in the right-hand column are called **index numbers** of profits and, in this case, January is known as the **base** month against which all others are compared.

The essentials of an index number then are that it illustrates **changes** by expressing the items in a time series as **percentages** of the item at a chosen **base** period.

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B. BUILDING UP AN INDEX NUMBER

Introduction

In commercial and economic affairs (and in some others, too) there are some very important quantities which are too complex to be measured directly; such things, for example, as the "level of industrial production" or the "cost of living". These are real enough things, but they are made up of a very large number of component parts, all affecting the main issue in different ways or to different extents. The index number notion is especially suited to dealing with such matters.

You should note that an index number is sometimes called an **index** (plural: **indices**).

Simple Index

Index numbers make a comparison between a value (quantity or price) in the current period and the corresponding value in a base period. All calculations are given in percentages without the % sign.

$$\text{Index number} = \frac{\text{Value}}{\text{Base Value}} \times 100$$

Example:

A litre of milk cost 800rwf in January 1995, 890rwf in 1998, 930rwf in 2000 and 960rwf in 2003

The milk price index in 1998 with 1995 as base is

$$\frac{890}{800} \times 100 = 111.25$$

Price Relatives

We can get round this problem by using the ratio of prices of a given item rather than the actual prices themselves. The price of a pint of milk in Year 10 as a percentage of its price in Year 1 is 420.0 and is called the **price relative** for milk in Year 10 (Year 1 = 100).

Similarly, we can work out price relatives for the other items.

(Remember, all we are doing is making the current price into a percentage of the base year price.)

Table 9.3

COMMODITY	PRICE RELATIVES IN YEAR 10 (YEAR 1 = 100)
Milk	$\frac{21}{5} \times 100 = 420.0$
Butter	$\frac{100}{40} \times 100 = 250.0$
Tea	$\frac{200}{60} \times 100 = 333.3$
Rice	$\frac{7}{2} \times 100 = 350.0$

From these price relatives we can now construct another index number called the **mean of relatives index**, which is just the arithmetic mean of the price relatives, i.e.

Mean of relatives index number for year 10 (Year 1 = 100)

$$\begin{aligned} &= \frac{420.0 + 250.0 + 333.3 + 350.0}{4} \\ &= \frac{1,353.3}{4} \\ &= 338.3 \end{aligned}$$

In other words, on this basis, prices in general appear to have risen 238% over the given period.

Another advantage of this price-relative type of index number is that the prices of all the commodities do not have to be in the same unit, although the prices of **each individual item** must be in the same units. This is a useful feature if you are dealing with results from different countries.

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C. WEIGHTED INDEX NUMBERS (LASPEYRES AND PAASCHE INDICES)

You may think that the mean of relatives index is still not very satisfactory, in that all items are treated as of equal importance and no account has been taken of the different quantities of the items consumed. For instance, the average family is much more concerned about a 5c increase in the price of a loaf of bread than a 10c increase in the price of a drum of pepper, as far more bread is consumed than pepper.

If you look back at Table 9.2, you will see that we are, in fact, given the average weekly consumption of each item in Year 1 and Year 10. You can see that the consumption pattern, as well as the prices, has changed over the 10-year period. We are interested in calculating an index for prices, so we have to be careful not to over-emphasise the increase in prices by incorporating the changes in consumption.

Weighted Aggregative Index Numbers

We can adopt either of two approaches:

- a) We can consider the consumption pattern in Year 1 as "typical" and:
 - (i) work out the total expenditure on the four items in Year 1; then,
 - (ii) work out what the total expenditure would have been in Year 10 if the family had consumed at Year 1 levels; and finally,
 - (iii) express the sum in (ii) as a percentage of (i) to form an index number.

This index is called a **base-weighted aggregative index** and in our example we work as follows:

Year 1 values are (Year 1 consumption x Year 1 prices)

Year 10 values are (Year 1 consumption x Year 10 prices)

In other words, we assume the consumption has not changed, only the prices.

The resulting table of values is:

Table 9.4

	Year 1	Year 10
Item	Expenditure Using Year 1 Consumption	Expenditure Using Year 1 Consumption
Milk	40	168
Butter	40	100
Tea	15	50
Rice	16	56
Total	111	374

Base-weighted aggregative index of prices in Year 10 (Year 1 = 100)

$$\begin{aligned}
 &= \frac{374}{111} \times 100 \\
 &= \mathbf{336.9}
 \end{aligned}$$

This type of index, where the weights are derived from quantities or values consumed in the base period, is known as a **Laspeyres index**, after the 19th century economist of that name.

The main defect of a Laspeyres index is that the weights become out-of-date as the pattern of demand changes. A Laspeyres index tends to **overstate** the change in prices, as it takes no account of the fall in consumption when prices rise.

b) The alternative method is to regard Year 10 as typical and to work all the figures as before, except that this time assume Year 10 consumption in Year 1.

This index is called the **current-weighted aggregative index**. For our example we have:

Table 9.5

	Year 1	Year 10
Item	Expenditure Using Year 10 Consumption	Expenditure Using Year 10 Consumption
Milk	25	105
Butter	20	50
Tea	7.5	25
Rice	12	42
Total	64.5	222

Current-weighted aggregative index of prices in Year 10 (Year 1 = 100)

$$= \frac{222}{64.5} \times 100$$

$$= 344.2$$

This type of index, where the weights are derived from quantities or values consumed in the current period, is known as a **Paasche index** after the 19th-century economist of that name.

The main defect of a Paasche index is that new weights have to be ascertained each time the index is calculated, and this involves time-consuming and expensive survey work. A Paasche index tends to **understate** the changes in prices, as most people tend to buy less of those commodities which have gone up in price.

Weighted Price-Relative Index Numbers

We can also form base-weighted or current-weighted price-relative index numbers. As before, we work out the price relatives for each commodity and as we now want to take into account the relative importance of each item in the family budget, we use as weight the actual expenditure on each item. The expenditure is used rather than the quantities consumed, to avoid exaggeration of variations arising from the change in consumption pattern rather than the change in price

- a) Base-Weighted Price-Relative Index Number (Laspeyres)

Table 9.6

Item	Price Relative	Expenditure in Year 1 (Weight) cent	Price Relative x Weight (cent)
Milk	420.0	40	16,800
Butter	250.0	40	10,000
Tea	333.3	15	5,000
Rice	350.0	16	5,600
	Total	111	37,400

Base-weighted price-relative index for Year 10 (Year 1 = 100)

$$\begin{aligned} &= \frac{\Sigma(\text{Price relative} \times \text{Weight})}{\Sigma \text{Weights}} \\ &= \frac{37,400}{111} \\ &= 336.9 \end{aligned}$$

b) Current-Weighted Price-Relative Index Number (Paasche)

Table 9.7

Item	Price Relative	Expenditure in Year 1 (Weight) cent	Price Relative x Weight (cent)
Milk	420.0	105	44,100
Butter	250.0	50	12,500
Tea	333.3	25	8,333
Rice	350.0	42	14,700
	Total	222	79,633

Current-weighted price-relative index for year 10 = (Year 1 = 100)

$$= \frac{\sum(\text{Price relative} \times \text{Weight})}{\sum(\text{Weights})}$$

$$= \frac{79,633}{222}$$

$$= 358.7$$

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D. FORMULAE

It will be useful at this stage to summarise our results so far by using formulae.

We use the normal notation:

p_0 = base year price

p_1 = current year price

q_0 = base year quantity

q_1 = current year quantity

n = number of commodities considered.

We have the following results:

$$(1) \quad \text{Price relative for current year} = \frac{\text{Current price}}{\text{Base price}} \times 100$$

$$= \frac{P_1}{P_0} \times 100$$

$$(2) \quad \text{Simple aggregative price index} = \frac{\sum \text{Current Price}}{\sum \text{Base Price}} \times 100$$

$$= \frac{\sum P_1}{\sum P_0} \times 100$$

$$(3) \quad \text{Simple mean of price relatives} = \frac{\sum \text{price relatives}}{N}$$

$$(4) \quad \text{Base-weighted aggregative price index (Laspeyres)}$$

$$= \frac{\sum (\text{Current price} \times \text{Base quantity})}{\sum \text{Base price} \times \text{Base quantity}} \times 100$$

$$= \frac{\sum (p_1 \cdot q_0)}{\sum (p_0 q_0)} \times 100$$

(5) Current-weighted aggregative price index (Paasche)

$$= \frac{\sum (\text{Current price} \times \text{Current quantity})}{\sum \text{Base price} \times \text{Current quantity}} \times 100$$

$$= \frac{\sum (p_1 \cdot p_0)}{\sum (p_0 q_1)} \times 100$$

(6) Weighted price-relative index = $\frac{\sum (\text{Price relatives} \times \text{Weight})}{\sum \text{Weight}}$

In (6), for a base-weighted price-relative index use (base price x base quantity) as the weight. And, for a current-weighted price-relative index, use (current price x current quantity) as the weight.

In trying to remember these it is probably simplest to memorise the price-relative, Laspeyres and Paasche formulae and to deduce the others from their descriptive names.

E. QUANTITY OR VOLUME INDEX NUMBERS

You must not think that we are always concerned with price indices. Often we are interested in **volume** or **quantity** indices as, for instance, in the Index of Industrial production which seeks to measure the changes in volume of output in a whole range of industries over a period of time. We can calculate such quantity index numbers in exactly the same sort of way as we dealt with the price indices, for example:

Quantity relative of a commodity in current year relative to base year

$$= \frac{q_1}{q_2} \times 100$$

Base-weighted aggregative quantity index (Laspeyres)

$$= \frac{\sum (q_1 p_1)}{\sum (q_2 p_1)} \times 100$$

Base-weighted quantity-relative index (Paasche)

$$= \frac{\sum \left(\frac{q_1}{q_0} \times 100 \right) (p_0 q_0)}{\sum (p_0 q_0)}$$

NB. There is no need to memorise these as they are really the same formulae with quantity substituted for price.

Notes

- a) The **price** of a commodity is now used as the weight for an aggregative quantity index and the **expenditure** on that commodity is used as the weight for a quantity-relative index.

- b) It is usual, if we are considering the situation from a producer's point of view rather than the consumer's, to call the index numbers **volume** indices and $\Sigma(p_0q_0)$, for example, will be the total value of production in the base year.
- c) Remember that for any commodity at any one time:
 Value = Price x Volume (producer's view)
 Expenditure = Price x Quantity (consumer's view).

Worked Example

Table 9.8 shows Rwanda imports of steel from Kenya. Calculate a base-weighted price Laspeyres index for all types of steel for Year 3 (Year 1 = 100).

Table 9.8

Grade	Year 1		Year 3	
	Quantity (q_0) (000 tonnes)	Value (p_0q_0) (RWF)	Quantity (q_1) (000 tonnes)	Value (p_1q_1) (RWF)
Steel grade 1	90	300	180	650
Steel grade 2	70	250	10	30
Steel grade 3	180	550	240	650
Steel grade 4	90	250	80	230
Steel grade 5	40	100	100	250

As we are asked for a price index, we must first calculate the price per tonne for each grade of steel using:

$$\text{Value} = \text{Price} \times \text{Quantity}$$

$$\text{i.e. Price} = \frac{\text{Value}}{\text{Quantity}}$$

Table 9.9

Year 1 Price (p_0) (RWF000/tonne)	Year 3 Price (p_1) (RWF000/tonne)
3.33	3.61
3.57	3.00
3.06	2.71
2.78	2.88
2.50	2.50

We have now to decide whether to use an aggregative index or a price-relative index. We are asked to find a base-weighted index. Interestingly, we should obtain the same answer whichever method we choose. However, there is less calculation involved in this particular example if we choose an aggregative type, so this is the one we shall work first; we will try the other later.

Base-weighted aggregative price index for Year 3 (Year 1 = 100)

$$= \frac{\text{Total value at Year 3 prices and Year 1 quantities}}{\text{Total value at Year 1 prices and Year 1 quantities}} \times 100$$

We have the Year 1 values in column two of Table 9.8 so we need only sum that column to get the denominator of the expression: RWF1,450 million pounds.

The numerator is the sum of the product of column one (q_0) in Table 9.8 and column two (p_1) in Table 9.9.

Table 9.10

	Value (p_1q_0) (RWFm) at Year 3 prices, Year 1 quantities
	324.9
	210.0
	487.8
	259.2
	100
Total	1,381.9

$$\text{Index for Year 3} = \frac{1,381.9}{1,450} \times 100$$

$$= 95.3 \text{ to 1 decimal place}$$

Therefore, there was an overall **decrease** in prices of 4.7% over the period Year 1 to Year 3.

You can check that using the price-relative method gives the same results. You will need a column for the price relatives and a column for the price relatives weighted with the base-year values.

Table 9.11

$\frac{p_1}{p_0} \times 100$	$\frac{p_1}{p_0} (p_0q_0) \times 100$
108.4	32,520
84.0	21,000
88.6	48,730
103.6	25,900
100.0	10,000
Total	138,150

Base-weighted price-relative index for Year 3 (Year 1 = 100)

$$= \frac{138,150}{1,450}$$

= 95.3 to 1 decimal place as before.

You will see that this must be so by simplifying the base-weighted price-relative formula.

There is not an equivalent rule for the current-weighted indices though.

You will see that in index number calculations you will have lots of multiplication and division to do. It is time-consuming to have to use logs at every stage, so if you do have a calculator, particularly one with a memory, it will be of great benefit.

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F. THE CHAIN-BASE METHOD

In the chain-base method, the index for the current period is based on the last (i.e. the immediately preceding) period.

For example, if we are calculating an index for 2003, we use 2002 as the base year; then, when we come to calculate the index for 2004, we use 2003 as the base year; and so on. This system has the advantage that it is always up-to-date and it is easy to introduce new items or delete old ones gradually without much upset to the reliability of the index. Its disadvantage is that it cannot be used for making comparisons over long periods of time, as we are simply comparing each year with the immediately preceding year.

If we do need to make long-term comparisons when a chain-base index number is in use, then it is necessary to convert the indices from a **chain base** to a **fixed base**. The method of working is shown in the following two examples.

Example 1

The indices for Years 1, 2, 3, (Year 0 as base = 100) are:

Year 0	100
Year 1	104
Year 2	104
Year 3	109

We are required to convert these to a chain-base set of indices. The Year 0 index remains the same at 100: the Year 1 index (based on Year 0) is still 104: the Year 2 index (based on Year 1) is 100 because the two years have the same index; and the Year 3 index (based on Year 2) is $(109 \times 100)/104 = 105$.

Example 2

The following indices were arrived at by a chain-base method. Convert them to Year 5 as a fixed base.

Year 5	100
--------	-----

Year 6	106
Year 7	110
Year 8	95
Year 9	100

The Year 5 index remains at 100; the Year 6 index (based on Year 5) remains at 106; the Year 7 index (based on Year 6) is 110 and therefore the Year 7 index will always be 110/100 of the Year 6 index, no matter what base is used.. Now, the Year 6 index (based on Year 5) is 106, and so the Year 7 index (based on Year 5) is $(110 \times 106)/100 = 116.6$. Similarly, the Year 8 index will be 95/100 of the Year 7 index, no matter what base is used. And so the Year 8 index (based on Year 5) is $(95 \times 116.6)/100 = 110.8$. The Year 9 index (based on Year 8) is 100 and therefore there is no change from Year 8 to Year 9: the Year 9 index (based on Year 5) is consequently the same as the Year 8 index (based on Year 5), namely 110.8.

G. DEFLATION OF TIME SERIES

These days we are all very familiar with the term "inflation", as it impinges directly on our own lives to a greater or lesser extent, depending upon the country we live in.

We make such remarks as, "Things cost twice as much now as they did ten years ago!", "We will need at least a 10% wage rise to keep up with inflation!", and, "I don't know where my money is going, everything costs so much nowadays".

We are, of course, referring to the effects of a positive cost of living index. I am sure you will not remember a time when the cost of living actually fell.

A static index from one year to the next indicates no inflation and although in Rwanda we managed a 6% index figure not too long ago, we have not had a 0% index for many many years. There has therefore always, to all intents and purposes, been some degree of inflation in the economy.

As all workers naturally want at least to maintain their standard of living, they look for annual wage rises at least equal to the cost of living rise, measured by the index. Very often, they negotiate clauses into their wage agreements specifying a cost of living increase, or inflation-proof clause, without being specific or trying to anticipate the figure.

For the purposes of the examination, we are interested in how these vague statements can be measured. What we do is relate values backwards in the same manner as with indices.

What do we Mean by Deflation?

Let us look at some more statements.

- 1) "Compared to 1960, the RWF in your pocket in 1979 was worth RWF 0.40."
- 2) "As far as food is concerned it was worth only RWF 0.35."
- 3) "As far as milk is concerned it was worth only RWF 0.20."

These statements say that in 1979 you got:

- 1) (1) 40% (2) 35% (3) 20%

as much spending power with your Franc on these items, as you did in 1960.

In each of these statements a change in prices has been expressed in terms of a so-called change in the value of the RWF.

When we are dealing with a single commodity such as milk, we can see easily how the "value" of the RWF is obtained. In 1960 the price of a pint of milk was RWF 0.03. In 1979 the price of a pint of milk was RWF 0.15, hence the purchasing power of the "milk" RWF in 1979 was a fifth of its purchasing power in 1960. In general, the purchasing power of the RWF for a single item is the reciprocal of the appropriate price relative written as a proportion, not as a percentage.

In our example:

$$\text{the milk price-relative} = \frac{15}{3} = 5$$

therefore the purchasing power of the RWF for a single item

$$= \frac{1}{\text{Price - relative}}$$

$$= \text{RWF } \frac{1}{5}$$

$$= \text{RWF } 0.20$$

When we are dealing with a group of commodities such as food or with the whole range of goods and services, the purchasing power of the RWF is worked out as the reciprocal of an appropriate price index number, again expressed as a proportion, i.e. the index of the food group in the CPI (Consumer Price Index) or the CPI itself.

Another way of looking at the problem is to consider the effect of increased prices on wages. A wage earner is more concerned with how much his or her wage will buy than the absolute amount he or she earns. To calculate "real" wages we divide the cash wages by an appropriate price index, expressed as a proportion, i.e. we are seeing the effect of rising prices on every Rwandan Franc in the pocket. This process is known as deflating. In principle it is easy enough but it is often difficult to find an appropriate index to use. For example, the CPI excludes expenditure on income tax, Social Insurance contributions, etc. which also affect the wage earner's purchasing power.

Changing the Index Base-Year

To convert indices from an earlier to a later base year, divide all the indices by the index for the new base year. This is really a variation on the technique of chain-based indices except that we relate to one particular year rather than continuing to roll forward.

We also multiply by 100 to regain percentage values.

The following indices have a base year of 1965 = 100:

Table 9.12

Year	1970	1974	1978	1982
Index	115	126	142	165

We will now convert to base year 1970 = 100 by dividing each index by 115 (1970's index) and multiplying by 100. You will immediately notice that the 1970 index becomes 100 as intended.

Table 9.13

Year	1970	1974	1978	1982
Index	100	$\frac{126}{115} \times 100$ = 110	$\frac{142}{115} \times 100$ = 123	$\frac{165}{115} \times 100$ = 143

Also, the 1965 index becomes $\frac{100}{115} \times 100 = 87$.

Example

Table 9.14 shows the average weekly earnings of male workers (aged 21 and over) during the years 1970-78. Also shown is the value of the CPI for these years with 1962 as base period. Determine the "real" average weekly earnings over the period 1970-78.

Table 9.14

Year	1970	1971	1972	1973	1974	1975	1976	1977	1978
CPI (1962 = 100)	140.2	153.4	164.3	179.4	208.1	258.5	301.3	349.1	378.0
Earnings (RWF)	28.05	30.93	35.82	40.92	48.63	59.58	66.97	72.89	83.50

After calculation as above, we obtain:

Table 9.15

Year	1970	1971	1972	1973	1974	1975	1976	1977	1978
CPI (1970 = 100)	100	109.4	117.2	128.0	148.4	184.4	214.9	249.0	269.6
"Real" Earnings (RWF)	28.05	28.27	30.56	31.97	32.77	32.31	31.16	29.27	30.97

We thus see that, although from 1970 to 1978 the average weekly earnings had apparently jumped by RWF55.45, i.e. increased by almost 200%, the "real" purchasing power had increased by RWF2.92 or 10%.

With the increase in inflation in recent years, the public at large has become more aware of index numbers, e.g. index-linked pensions, savings, insurance premiums, etc. However, the public does not realise that index numbers are of necessity imperfect measures the values of which can be manipulated by changes in base year or by changes in the weighting system. For pensions, the decision has to be made whether to link them with earnings or with prices, and if with earnings the earnings of whom: manual workers, all workers, workers in the same industry? With house insurance premiums, is the index used to be based on the estimated market value of the house or on the cost of clearing the site and rebuilding the house? There is increasing discussion on these matters so do be on the look-out for such articles and relate them to your own knowledge of how index numbers are constructed.

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STUDY UNIT 10

Percentages and Ratios, Simple and Compound Interest, Discounted Cash Flow

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A. PERCENTAGES

A percentage is really a fraction where the denominator is 100, and we use the symbol %. To convert a fraction or a decimal to a percentage, simply multiply by 100.

$$\frac{7}{100} \text{ becomes } \frac{7}{100} \times 100 = \frac{700}{100} = 7\%$$

$$\frac{3}{10} \text{ becomes } \frac{3}{10} \times 100 = \frac{300}{10} = 30\%$$

$$0.8 \text{ becomes } 0.8 \times 100 = 80\%$$

$$0.15 \text{ becomes } 0.15 \times 100 = 15\%$$

$$\frac{1}{3} \text{ becomes } \frac{1}{3} \times 100 = \frac{100}{3} = 33\frac{1}{3}\%$$

To find the percentage of a number, we simply convert the percentage to a fraction and multiply:

$$8\% \text{ OF RWF}10,000 = \frac{8}{100} \times 10000 = \text{RWF}800$$

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B. RATIOS

Introduction

A ratio is another form of fraction just like a percentage, decimal or common fraction. It is a relationship between two numbers or two like values. Consider the following situation concerning a small town:

Employed	3,000
Unemployed	<u>1,000</u>
Total workforce	<u>4,000</u> (available for employment)

This can be expressed in several ways:

$\frac{1}{4}$ of the workforce is unemployed (fraction)

25% of the workforce is unemployed (percentage)

0.25 of the workforce is unemployed (decimal)

The same situation can also be expressed as a ratio. A ratio could say that 1 out of 4 of the workforce was unemployed, or alternatively, for every person unemployed, 3 persons were employed.

A special symbol is used for expressing a ratio. The colon sign (:) indicates the important relationship. Thus in terms of the foregoing example, the relationship is 1:4 or 1:3 depending on whether you wish to say:

"1 in every 4 is unemployed" or

"For every 1 unemployed, 3 are working".

Another way of expressing a ratio is by using the word "per":

30 kilometres per hour

60 words per minute

20 kilometres per day

2 inches per week

3 doses per day

5 meetings per year

Ratios are a particularly important part of the language of business and are used to express important relationships.

If you intend to proceed to more advanced accounting studies, a thorough understanding of ratios at this stage will provide a useful foundation for the systematic analysis of accounting information.

To Reduce the Ratio to its Lowest Terms

Put the first figure over the second figure and cancel the resulting fraction. Then re-express as a ratio in the form numerator : denominator, for example:

Calculate the ratio of 17 to 85.

$$\frac{17}{85} = \frac{1}{5}$$

The ratio of 17 to 85 is therefore 1:5 or 1 in 5.

Of course the ratio could be stated as 17:85 but by dividing 85 by 17 it is reduced to the lowest possible terms, and therefore made more manageable.

The ratio of 38:171 is better stated as 2:9.

It is important to realise that in its original state, the relationship between the two numbers is not incorrect. But by reducing the terms the relationship becomes a much easier one to follow. This can be seen from the following example:

What is the relationship of 411 to 137?

It could be correctly stated as 411:137 but see how much more meaningful the relationship is after the terms have been reduced.

$$\frac{411}{137} = 3$$

The ratio is therefore 3:1.

Divide a Quantity According to a Given Ratio

Add the terms of the ratio to find the total number of parts. Find what fraction each term of the ratio is to the whole. Divide the total quantity into parts according to the fraction. Here again, this is much simpler when actual figures are introduced:

- a) RWF60 has to be divided between 2 brothers in the ratio of 1:2. Ascertain the share of each brother.

$$\text{Total number of parts} = 1 + 2 = 3$$

$$\frac{60}{3} = \text{RWF20} = \text{one part}$$

therefore one brother gets RWF20 (one part), the other RWF40 (two parts).

- b) 80 books have to be divided between 4 libraries in the proportions 2:3:5:6. What does each library receive? Total no. of parts = $2 + 3 + 5 + 6 = 16$

$$\frac{80}{16} = 5 = \text{one part}$$

Therefore:

$$\text{library 1 gets } 2 \times 5 = 10 \text{ books}$$

$$\text{library 2 gets } 3 \times 5 = 15 \text{ books}$$

$$\text{library 3 gets } 5 \times 5 = 25 \text{ books and}$$

$$\text{library 4 gets } 6 \times 5 = \underline{30} \text{ books}$$

$$\underline{\underline{80}}$$

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C. SIMPLE INTEREST

Interest (I) is a charge for the use of money for a specific time. This charge is usually expressed as a percentage called the **rate per cent per annum**. Three factors determine the amount of interest:

- a) The sum of money on which the interest is payable; this is known as the **principal (P)**.
- b) The **rate (R)**.
- c) The length of **time (Y)** for which the money is borrowed.

When the interest due is added to the principal, the sum is called the **amount (A)**, which is the amount to be repaid.

Simple interest is interest reckoned on a fixed principal. Simple interest is, therefore, the same for each year, and the total is found by multiplying the interest for one year by the number of years.

Examples

- a) Find the simple interest on RWF200 for 3 years at 4% per annum.

$$\text{Simple interest} = \text{RWF}200 \times \frac{4}{100} \times 3 = \text{RWF}24.$$

- b) Find the simple interest on RWF200 for 3 months at 4% per annum.

$$\text{Simple interest} = \text{RWF}200 \times \frac{4}{100} \times \frac{3}{12} = \text{RWF}2.$$

To calculate the simple interest on a sum of money lent for a given time at a given rate per cent per annum, the following formula is used:

$$I = \frac{P \times R \times Y}{100} \text{ (Y = time in years).}$$

When using symbols, the multiplication sign should be omitted:

$I = \frac{PRY}{100}$

(You should memorise this formula.)

You can use this formula to solve any problem in which we are required to find the principal, rate, time or interest. There are four quantities involved, so given any three, we can find the other one.

$$I = \frac{PRY}{100}$$

Multiplying both sides by 100:

$$I \times 100 = PRY$$

$$P = \frac{I \times 100}{YR}$$

$$Y = \frac{I \times 100}{PR}$$

$$R = \frac{I \times 100}{PY}$$

$$\text{Unknown factor} = \frac{\text{Interest} \times 100}{\text{Two known factors}}$$

When you are working examples always:

- State the formula.
- Give the value to be substituted.
- See that the numbers you use are in the correct units.
- Remember to write the correct unit against the answer, not just a number only.

Examples

- a) At what rate of simple interest will RWF500 earn RWF75 in 4 years?

$$R = \frac{I \times 100}{PY}$$

$$I = \text{RWF}75, \quad P = \text{RWF}500, \quad Y = 4$$

$$R = \frac{75 \times 100}{500 \times 4} = 3\frac{3}{4}\%$$

- b) What sum of money will amount to RWF500 in 4 years at 4% per annum simple interest?

$$A = P + \frac{PYR}{100} = P \frac{100 + YR}{100}$$

$$\therefore P = \frac{100A}{100 + YR}$$

$$A = 500, \quad Y = 4, \quad R = 4$$

$$P = \frac{100 \times 500}{100 + 4 \times 4} = \text{RWF}431$$

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D. COMPOUND INTEREST

Definition

In compound interest, the interest due is added to the principal at stated intervals, and interest is reckoned on this increased principal for the next period, and so on, the principal being increased at each period by the amount of interest then due.

Example

Find the compound interest and the simple interest on RWF1,000 invested at $2\frac{1}{2}\%$ per annum for 4 years.

	RWF
a) Principal	1,000
Add 1st year's interest at $2\frac{1}{2}\%$	<u>25</u>
Amount at end of 1st year	1,025
Add 2nd year's interest at $2\frac{1}{2}\%$	<u>25.625</u>
Amount at end of year 2	1,050.625
Add 3rd year's interest at $2\frac{1}{2}\%$	<u>26.265625</u>
Amount at end of year 3	1,076.890625
Add 4th year's interest at $2\frac{1}{2}\%$	<u>26.922266</u>
Final amount at end of 4 years	1,103.812891
	= <u>RWF1,103.81.</u>

Therefore, compound interest

$$\begin{aligned} &= \text{Final amount} - \text{Principal} \\ &= \text{RWF1,103.81} - \text{RWF1,000} \\ &= \text{RWF103.81.} \end{aligned}$$

- b) The simple interest on RWF1,000 at 2 ½ % per annum over 4 years is RWF25 per annum (always constant) or RWF100,

$$1,000 \times \frac{2.5}{100} \times 4 = \text{RWF}100$$

Now work through the above example by yourself to ensure that you fully understand the principle involved, and then answer the following question.

Compound Interest Formula

If P is the principal, and if r is the rate of interest on RWF1 for 1 year, then the interest on P for 1 year is P x r, written as Pr.

At the end of the first year the interest is added to the principal; therefore the new principal = P + Pr or P(1 + r).

At the end of the second year the interest on the new principal, i.e. P(1 + r), is P(1 + r) x r or Pr(1 + r).

The principal at the end of the second year is now P(1 + r) + Pr(1 + r). This can be written as (P + Pr) (1 + r) which equals P(1 + r)².

You will see that the new principal at the end of n years is equal to P(1 + r)ⁿ, and we therefore have the formula for the evaluation of compound interest, which is:

$$A = P(1 + r)^n$$

Where A = Final amount

P = Original sum invested

r = Rate of interest per annum on RWF1

n = Number of years.

Remember that r is the rate of interest on RWF1 for 1 year. Therefore, if the question refers to a rate of interest of 5 **per RWF** per annum,

$$(1 + r) \text{ becomes } 1 + \frac{5}{100} = 1.05.$$

You must become accustomed to thinking in these terms, so that visualising the formula becomes automatic. Learn the formula by heart. Say it to yourself over and over again until it is firmly imprinted on your mind.

Now that you have learned and understood the formula $A = P(1 + r)^n$, it is evident that the real problem lies in the evaluation of $(1 + r)^n$. This is most conveniently done by the use of a calculator or by logarithms, especially when n is large. It is usual to use seven-figure logarithms, as four-figure tables are not sufficiently accurate for compound interest calculations. However, with the time constraints that examinations impose, a calculator is preferable.

Example

Calculate the compound interest to the nearest cent on RWF1,000 for 2 years at 6% per annum, interest being calculated each six months.

In this case, n is the number of six-month periods (not years) and we must adjust the interest rate accordingly; so 6% p.a. = 3% per half year.

$$\begin{aligned} A &= P(1 + r)^n \\ &= 1,000 (1.03)^4 \\ &= 1,000 (1.03 \times 1.03 \times 1.03 \times 1.03) \\ &= 1,000 \times 1.12550881 \\ &= 1,125.50881 \\ &= \text{RWF}1,125.51 \end{aligned}$$

$$\begin{aligned}
\text{Therefore, Compound interest} &= A - P \\
&= \text{RWF}1,125.51 - \text{RWF}1,000 \\
&= \text{RWF}125.51 \text{ to the nearest RWF.}
\end{aligned}$$

Having worked carefully through the preceding examples, try to answer the following question.

Additional Investment

Suppose that you decide to invest RWF2,000 at the beginning of a particular year and that you add RWF100 to this investment at the end of each year. If interest is compounded at 9% per annum, then we can deduce:

The amount invested at the end of the first year is

$$\text{RWF}2,000(1 + 0.09) + \text{RWF}100.$$

The amount invested at the end of the second year is

$$\text{RWF}2,000(1 + 0.09)^2 + \text{RWF}100(1 + 0.09) + 100.$$

The amount invested at the end of the nth year is

$$\begin{aligned}
&\text{RWF}2,000(1 + 0.09)^n + \text{RWF}100(1 + 0.09)^{n-1} + \text{RWF}100(1 + 0.09)^{n-2} + \dots + \\
&\text{RWF}100(1 + 0.09) + \text{RWF}100.
\end{aligned}$$

Ignoring the first term on the right-hand side, the other terms can be written:

$$\begin{aligned}
&\text{RWF}100 + \text{RWF}100(1 + 0.09) + \dots + \text{RWF}100(1 + 0.09)^{n-2} \\
&\qquad\qquad\qquad + \text{RWF}100(1 + 0.09)^{n-1}
\end{aligned}$$

This expression is called a geometric progression. The first term is 100 and each successive term is multiplied by $(1 + 0.09)$. This factor of $(1 + 0.09)$ is called the "common ratio". There is a formula for the sum of such expressions. In this case it is:

$$RWF100 \frac{(1.09^n - 1)}{1.09 - 1} = RWF100 \frac{(1.09^n - 1)}{0.09}$$

Supposing we wish to know the amount invested after 3 years, then we put $n = 3$.

$$\begin{aligned} \text{Amount} &= RWF2,000(1.09)^3 + RWF100 \frac{(1.09^3 - 1)}{0.09} \\ &= RWF2,590.06 + RWF327.81 \\ &= RWF2,917.87. \end{aligned}$$

Supposing we wish to know the amount invested after 3 years, then we put $n = 3$.

$$\begin{aligned} \text{Amount} &= RWF2,000(1.09)^3 + RWF100 \frac{(1.09^3 - 1)}{0.09} \\ &= RWF2,590.06 + RWF327.81 \\ &= RWF2,917.87. \end{aligned}$$

In general, if an amount P is invested at the beginning of a year and a further amount a is invested at the **end** of each year, then the sum, S , invested after n years is:

$$\begin{aligned} S &= P(1+r)^n + a(1+r)^{n-1} + a(1+r)^{n-2} + \dots + a(1+r) + a \\ &= P(1+r)^n + a \frac{(1+r)^n - 1}{(1+r) - 1} \\ &= P(1+r)^n + a \frac{(1+r)^n - 1}{r} \end{aligned}$$

$$S = \left(P + \frac{a}{r}\right)(1+r)^n - \frac{a}{r}$$

We have not attempted to prove that this formula is correct, but have simply stated it. Any proof is outside the scope of this course, but can be found in books, if you are sufficiently interested.

E. INTRODUCTION TO DISCOUNTED CASH FLOW PROBLEMS

If a business is to continue earning profit, its management should always be alive to the need to replace or augment fixed assets. This usually involves investing money (capital expenditure) for long periods. The longer the period the greater is the uncertainty and, therefore, the risk involved. With the advent of automation, machinery, equipment and other fixed assets have tended to become more complex and costly. Careful selection of projects has never been so important. One method of selecting the most profitable investments follows.

These techniques do not replace judgement and the other qualities required for making decisions. However, it is true to say that the more information available, the better able a manager is to understand a problem and reach a rational decision.

Classification of Investment Problems

Capital investment problems may be classified into the following types, and each is amenable to discounted cash flow analysis.

- a) The replacement of, or improvement in, existing assets by more efficient plant and equipment (often measured by the estimated cost savings).
- b) The expansion of business facilities to produce and market new products (measured by the forecast of additional profitability against the proposed capital investment).
- c) Decisions regarding the choice between alternatives where there is more than one way of achieving the desired result.
- d) Decisions whether to purchase or lease assets.

Basis of the Method

The method is based on the criterion that the total present value of all increments of income from a project should, when calculated at a suitable rate of return on capital, be at least sufficient to cover the total capital cost. It takes account of the fact that the earlier the return the more valuable it is, for it can be invested to earn further income meanwhile.

By deciding on a satisfactory rate of return for a business, this can then be applied to several projects over their total life to see which gives the best present cash value.

For any capital investment to be worthwhile, it must give a return sufficient to cover the initial cost and also a fair income on the investment. The rate which will be regarded as "fair income" will vary with different types of business, but as a general rule it should certainly be higher than could be obtained by an equivalent investment in shares.

Information Required

To make use of DCF we must have accurate information on a number of points. The method can only be as accurate as the information which is supplied.

The following are necessary as a basis for calculation:

- a) Estimated cash expenditure on the capital project.
- b) Estimated cash expenditure over each year.
- c) Estimated receipts each year, including scrap or sale value, if any, at the end of the asset's life.
- d) The life of the asset.
- e) The rate of return expected (in some cases you will be given a figure for "cost of capital" and you can easily use this rate in the same way to see whether the investment is justified).

The **cash flow** each year is the actual amount of cash which the business receives or pays each year in respect of the particular project or asset (a net figure is used). This represents the difference between (c) and (b).

Clearly the receipts and expenditures may occur at irregular intervals throughout the year, but calculations on this basis would be excessively complicated for problems such as may arise in your examination. So, unless you are told otherwise, you can assume that the net receipt or expenditure for the year occurs at the end of the relevant year

Importance of "Present Value"

Before we proceed to a detailed examination of the method used by DCF there is one important concept which you need to understand - the idea of **present value**.

Let us take a businessman who is buying a machine. It will give him, let us say, an output worth RWF100 at the end of the first year, and the same at the end of each successive year. He must bear this in mind when buying the machine which costs, say, RWF1,000. But he must pay out the RWF1,000 now. His income, on the other hand, is not worth its full value now, because it will be a year before he will receive the first RWF100, two years before he will receive the second RWF100, and so on. So if we think of the present value of the income which he is to receive, the first RWF100 is really worth less than RWF100 **now**, and the second RWF100 is worth less still. In fact, the present value of each increment of RWF100 is the sum now which, at compound interest, will represent RWF100 when the sum falls due.

This can easily be calculated, or ascertained from specially prepared **present value tables**, which take account of time and of varying interest rates (see Tables (a)-(d)).

These tables are easily used. We can see, for example, that if we assume a cost of capital of 7%, RWF1 in two years' time is worth RWF0.8734 **now**. This is the sum which would grow to RWF1 in two years at compound interest of 7%. Thus we have established the present value of RWF1 in two years' time, discounted at compound interest of 7%.

We can now look again at the businessman and his machine. We will assume the cost of capital is also 7%. The present value of the first year's income (received at the end of the year, for the purposes of this example) is $100 \times \text{RWF}0.9346$ and the present value of the second year's income is $100 \times \text{RWF}0.8734$. The same method can be used for succeeding years in the same way.

An extract from the present value tables will usually be given with examination questions requiring calculations.

Table 10.1: Present Value of RWF 1 (to 4 sig. figs)

Rate Year	2%	3%	4%	5%	6%	7%
1	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346
2	0.9612	0.9426	0.9246	0.9070	0.8900	0.8734
3	0.9423	0.9151	0.8890	0.8638	0.8396	0.8163
4	0.9238	0.8885	0.8548	0.8227	0.7921	0.7629
5	0.9057	0.8626	0.8219	0.7835	0.7473	0.7130
6	0.8880	0.8375	0.7903	0.7462	0.7050	0.6663
7	0.8706	0.8131	0.7599	0.7107	0.6651	0.6227
8	0.8535	0.7894	0.7307	0.6768	0.6274	0.5820
9	0.8368	0.7664	0.7026	0.6446	0.5919	0.5439
10	0.8203	0.7441	0.6756	0.6139	0.5584	0.5083
11	0.8043	0.7224	0.6496	0.5847	0.5268	0.4751
12	0.7885	0.7014	0.6246	0.5568	0.4970	0.4440
13	0.7730	0.6810	0.6006	0.5303	0.4688	0.4150
14	0.7579	0.6611	0.5775	0.5051	0.4473	0.3878
15	0.7430	0.6419	0.5553	0.4810	0.4173	0.3624
16	0.7284	0.6232	0.5339	0.4581	0.3936	0.3387
17	0.7142	0.6050	0.5135	0.4363	0.3714	0.3166
18	0.7002	0.5874	0.4936	0.4155	0.3503	0.2959

Table 10.2: Present Value of RWF 1 (to 4 sig. figs) (Contd)

Rate Year	8%	9%	10%	11%	12%	13%
1	0.9259	0.9174	0.9091	0.9009	0.8929	0.8850
2	0.8573	0.8417	0.8264	0.8116	0.7972	0.7831
3	0.7938	0.7722	0.7513	0.7312	0.7118	0.6931
4	0.7350	0.7084	0.6830	0.6587	0.6355	0.6133
5	0.6806	0.6499	0.6209	0.5935	0.5674	0.5428
6	0.6302	0.5963	0.5645	0.5346	0.5066	0.4803
7	0.5835	0.5470	0.5132	0.4817	0.4523	0.4251
8	0.5403	0.5019	0.4665	0.4339	0.4039	0.3762
9	0.5002	0.4604	0.4241	0.3909	0.3606	0.3329
10	0.4632	0.4224	0.3855	0.3522	0.3220	0.2946
11	0.4289	0.3875	0.3505	0.3173	0.2875	0.2607
12	0.3971	0.3555	0.3186	0.2858	0.2567	0.2307
13	0.3677	0.3262	0.2897	0.2575	0.2292	0.2042
14	0.3405	0.2992	0.2633	0.2320	0.2046	0.1807
15	0.3152	0.2745	0.2394	0.2090	0.1827	0.1599
16	0.2919	0.2519	0.2176	0.1883	0.1631	0.1415
17	0.2703	0.2311	0.1978	0.1696	0.1456	0.1252
18	0.2502	0.2120	0.1799	0.1528	0.1300	0.1108

Table 10.3: Present Value of RWF 1 (to 4 sig. figs) (Contd)

Year \ Rate	14%	15%	16%	17%	18%	19%
1	0.8763	0.8696	0.8621	0.8547	0.8475	0.8403
2	0.7693	0.7617	0.7544	0.7472	0.7402	0.7331
3	0.6750	0.6675	0.6602	0.6531	0.6461	0.6391
4	0.5921	0.5848	0.5776	0.5705	0.5635	0.5565
5	0.5194	0.5122	0.5051	0.4981	0.4911	0.4841
6	0.4556	0.4485	0.4415	0.4345	0.4275	0.4205
7	0.3996	0.3926	0.3856	0.3786	0.3716	0.3646
8	0.3506	0.3436	0.3366	0.3296	0.3226	0.3156
9	0.3075	0.3005	0.2935	0.2865	0.2795	0.2725
10	0.2697	0.2627	0.2557	0.2487	0.2417	0.2347
11	0.2366	0.2296	0.2226	0.2156	0.2086	0.2016
12	0.2076	0.2006	0.1936	0.1866	0.1796	0.1726
13	0.1821	0.1751	0.1681	0.1611	0.1541	0.1471
14	0.1597	0.1527	0.1457	0.1387	0.1317	0.1247
15	0.1401	0.1331	0.1261	0.1191	0.1121	0.1051
16	0.1229	0.1159	0.1089	0.1019	0.0949	0.0879
17	0.1078	0.1008	0.0938	0.0868	0.0798	0.0728
18	0.09456	0.08756	0.08056	0.07356	0.06656	0.05956

Table 10.4: Present Value of RWF 1 (to 4 sig. figs) (Contd)

Rate Year	20%	21%	22%	23%	24%	25%
1	0.8333	0.8264	0.8197	0.8130	0.8065	0.8000
2	0.6944	0.6830	0.6719	0.6610	0.6504	0.6400
3	0.5787	0.5645	0.5507	0.5374	0.5245	0.5120
4	0.4823	0.4665	0.4514	0.4369	0.4230	0.4096
5	0.4019	0.3855	0.3700	0.3552	0.3411	0.3277
6	0.3349	0.3186	0.3033	0.2888	0.2751	0.2621
7	0.2791	0.2633	0.2486	0.2348	0.2218	0.2097
8	0.2326	0.2176	0.2038	0.1909	0.1789	0.1678
9	0.1938	0.1799	0.1670	0.1522	0.1443	0.1342
10	0.1615	0.1486	0.1369	0.1262	0.1164	0.1074
11	0.1346	0.1228	0.1122	0.1026	0.09383	0.08590
12	0.1122	0.1015	0.09198	0.08339	0.07567	0.06872
13	0.09346	0.08391	0.07539	0.06780	0.06103	0.05498
14	0.07789	0.06934	0.06180	0.05512	0.04921	0.04398
15	0.06491	0.05731	0.05065	0.04481	0.03969	0.03518
16	0.05409	0.04736	0.04152	0.03643	0.03201	0.02815
17	0.04507	0.03914	0.03403	0.02962	0.02581	0.02252
18	0.03756	0.03235	0.02789	0.02408	0.02082	0.01801

Procedure

Since our DCF appraisal will be carried out before the beginning of a project, we shall have to reduce each of the net receipts/expenditures for future years to a present value. This is "discounting" the cash flow, which gives DCF its name, and it is usually done by means of tables, an extract of which you have already seen. You should remember, incidentally, that at the very start of a project the capital expenditure itself may be made, so that at that point there may be a substantial "negative" present value, since money has been paid out and nothing received. If all the present values of the years of the life of the investment (including the original cost) are added together, the result will be the net present value. This is known as the NPV and is a vital factor, because if it is positive it shows that the discounted receipts are greater than expenditures on the project, so that at that rate of interest the project is proving more remunerative than the stated interest rate. The greater the NPV the greater the advantages of investing in the project rather than leaving the money at the stated rate of interest. But if the NPV is a minus quantity, it shows that the project is giving less return than would be obtained by investing the money at that rate of interest.

A practical example will probably be helpful at this point.

Example

A businessman is considering the purchase of a machine costing RWF1,000, which has a life of 3 years. He calculates that during each year it will provide a net receipt of RWF300; it will also have a final scrap value of RWF200. Alternatively, he could invest his RWF1,000 at 6%. Which course would be more advantageous?

First we must work out the cash flow:

	Receipts	Payments	Net Receipts
Year 0	Nil	RWF1,000	- RWF1,000
Year 1	RWF300	Nil	+ RWF300
Year 2	RWF300	Nil	+ RWF300
Year 3	RWF500	Nil	+ RWF500

(Remember that the scrap value will count as a receipt at the end of the third year.)

But the businessman could be earning 6% interest instead; so this is the cost of his capital, and we must now discount these figures to find the present value. We can use the extract from the tables which we have already seen.

	Net Receipts	Discount Factor	Present Factor
Year 0	- RWF1,000	1.0000	- RWF1,000.00
Year 1	+ RWF300	0.9434	+ RWF283.02
Year 2	+ RWF300	0.8900	+ RWF267.00
Year 3	+ RWF500	0.8396	+ RWF419.80
		Net present value	- RWF30.18

As we have seen above, a negative NPV means that the investment is not profitable at that rate of interest. So the businessman would lose by putting his money into the machine. The best advice is for him to invest at 6%.

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F. TWO BASIC DCF METHODS

You have now seen a simple example of how DCF is used, and you already have a basic knowledge of the principles which the technique employs. There are two different ways of using DCF - the yield (or rate of return) method, and the net present value method, which was used in the above example.

The important point to remember is that both these methods give identical results. The difference between them is simply the way they are used in practice, as each provides an easier way of solving its own particular type of problem.

As you will shortly see, the yield method involves a certain amount of trial-and-error calculation. Questions on either type are possible, and you must be able to distinguish between the methods and to decide which is called for in a particular set of circumstances.

In both types of calculation there is the same need for accurate information as to cash flow, which includes the initial cost of a project, its net income or outgoings for each year of its life, and the final scrap value of any machinery.

Yield (Internal Rate of Return) Method

This method is used to find the yield, or rate of return, on a particular investment. By "yield" we mean the percentage of profit per year of its life in relation to the capital employed. In other words, we must allow for **repayment of capital** before we consider income as being profit for this purpose. The profit may vary over the years of the life of a project, and so may the capital employed, so an **average** figure needs to be produced.

DCF, by its very nature, takes all these factors into account.

The primary use of the method is to evaluate a particular investment possibility against a guideline for yield which has been laid down by the company concerned. For example, a company may rule that investment may only be undertaken if a 10% yield is obtainable. We then have to see whether the yield on the desired investment measures up to this criterion. In another case, a company may simply wish to know what rate of return is obtainable from a

particular investment; thus, if a rate of 9% is obtainable, and the company's cost of capital is estimated at 7%, it is worth its while to undertake the investment.

What we are trying to find in assessing the figures for a project is the yield which its profits give in relation to its cost. We want to find the exact rate at which it would be breaking even, i.e. the rate at which discounted future cash flow will exactly equal the present cost, giving an NPV of 0. Thus if the rate of return is found to be 8%, this is the rate at which it is equally profitable to undertake the investment or not to undertake it; the NPV is 0. Having found this rate, we know that if the cost of capital is above 8%, the investment will be unprofitable, whereas if it is less than 8%, the investment will show a profit. We thus reach the important conclusion that once we have assembled all the information about a project, the yield, or rate of return, will be the rate which, when used to discount future increments of income, will give an NPV of 0. We shall then know that we have found the correct yield.

You should ensure that you know exactly how and when to use the method, as practical questions are very much more likely than theoretical ones in the examination.

a) When to Use the Yield Method

This is not a difficult problem, because you will use the method whenever you require to know the rate of return, or yield, which certain increments of income represent on capital employed. You must judge carefully from any DCF question whether this is what you need to know.

b) How to Use the Yield Method

The calculation is largely dependent on trial and error. When you use this method, you know already that you are trying to find the rate which, when used to discount the various increments of income, will give an NPV of 0. You can do this only by trying out a number of different rates until you hit on the correct result. A positive NPV means that the rate being tried is lower than the real rate; conversely, a negative NPV means that too high a rate is being used. So you need to work the problem out as many times as is necessary to hit on the appropriate rate for obtaining the NPV of 0. If this process is done sensibly, for simple problems such as those which we are going to encounter, it should not take many steps to hit upon the right result. Watch out for any instructions concerning "rounding" of yields - for example, "to the nearest $\frac{1}{2}$ %".

Example

A businessman is considering investment in a project with a life of 3 years, which will bring a net income in the first, second and third years of RWF800, RWF1,000 and RWF1,200 respectively. The initial cost is RWF2,500 and there will be no rebate from scrap values at the end of the period. He wishes to know, to the nearest 1%, the yield which this would represent. Using the present value tables given earlier, make the necessary calculation.

We must begin by choosing a possible rate, and testing to see how near this is. Let us try 7%. Referring to the tables, we reach the following results:

Table 10.5

Year	Net Income/ Outgoings	Discount Factor	Discounted Present Value
0	- RWF2,500	1.0000	- RWF2,500.00
1	+ RWF800	0.9346	+ RWF747.68
2	+ RWF1,000	0.8734	+ RWF873.40
3	+ RWF1,200	0.8163	+ <u>RWF979.56</u>
		Net present value	<u>+ RWF100.64</u>

A positive NPV, as we have seen, means that we have taken too low a rate for our attempt. Let us try 10% instead:

Table 10.6

Year	Net Income/ Outgoings	Discount Factor	Discounted Present Value
0	- RWF2,500	1.0000	- RWF2,500.00
1	+ RWF800	0.9091	+ RWF727.28
2	+ RWF1,000	0.8264	+ RWF826.40
3	+ RWF1,200	0.7513	+ <u>RWF901.56</u>
		Net present value	<u>- RWF44.76</u>

This time we have obtained a negative NPV so our rate of 10% must be too high. We now know that the rate must be between 7% and 10%. Only a proper calculation can give us the true answer, but having obtained a positive NPV for 7% and a negative NPV for 10%, the approximate rate can be ascertained by interpolation using the formula:

$$\text{Rate} = X + \frac{a}{a + b}(Y - X)$$

where:

- X = Lower rate of interest used
- Y = Higher rate of interest used
- a = Difference between the present values of the outflow and the inflow at X%
- b = Difference between the present values of the outflow and the inflow at Y%

We can extend the trial and error technique as follows.

+ RWF100 is further from zero than - RWF44

so, 7% is further from zero NPV than 10%.

So we shall try 9%.

Table 10.7

Year	Net Income/ Outgoings	Discount Factor	Discounted Present Value
0	- RWF2,500	1.0000	- RWF2,500.00
1	+ RWF800	0.9174	+ RWF733.92
2	+ RWF1,000	0.8417	+ RWF841.70
3	+ RWF1,200	0.7722	+ RWF926.64
		Net present value	+ RWF2.26

Clearly, since we are working to the nearest 1% we are not going to get any closer than this. However, if you have time available, there is no reason why you should not check the next nearest rate (in this case, 8%) just to check that you already have the nearest one.

So the yield from this investment would be 9%.

Alternatively, interpolation may be performed graphically rather than by calculation, as shown in Figure 10.1. The discount rate is on the horizontal axis and the net present value on the vertical axis. For each of the two discount rates, 7% and 10%, we plot the corresponding net present value. We join the two points with a ruled line. The net present value is zero where this line crosses the horizontal axis. The discount rate at this point is the required internal rate of return. From Figure 10.1 we see that the rate is 9% correct to the nearest 1%, and this confirms the result of the calculation.

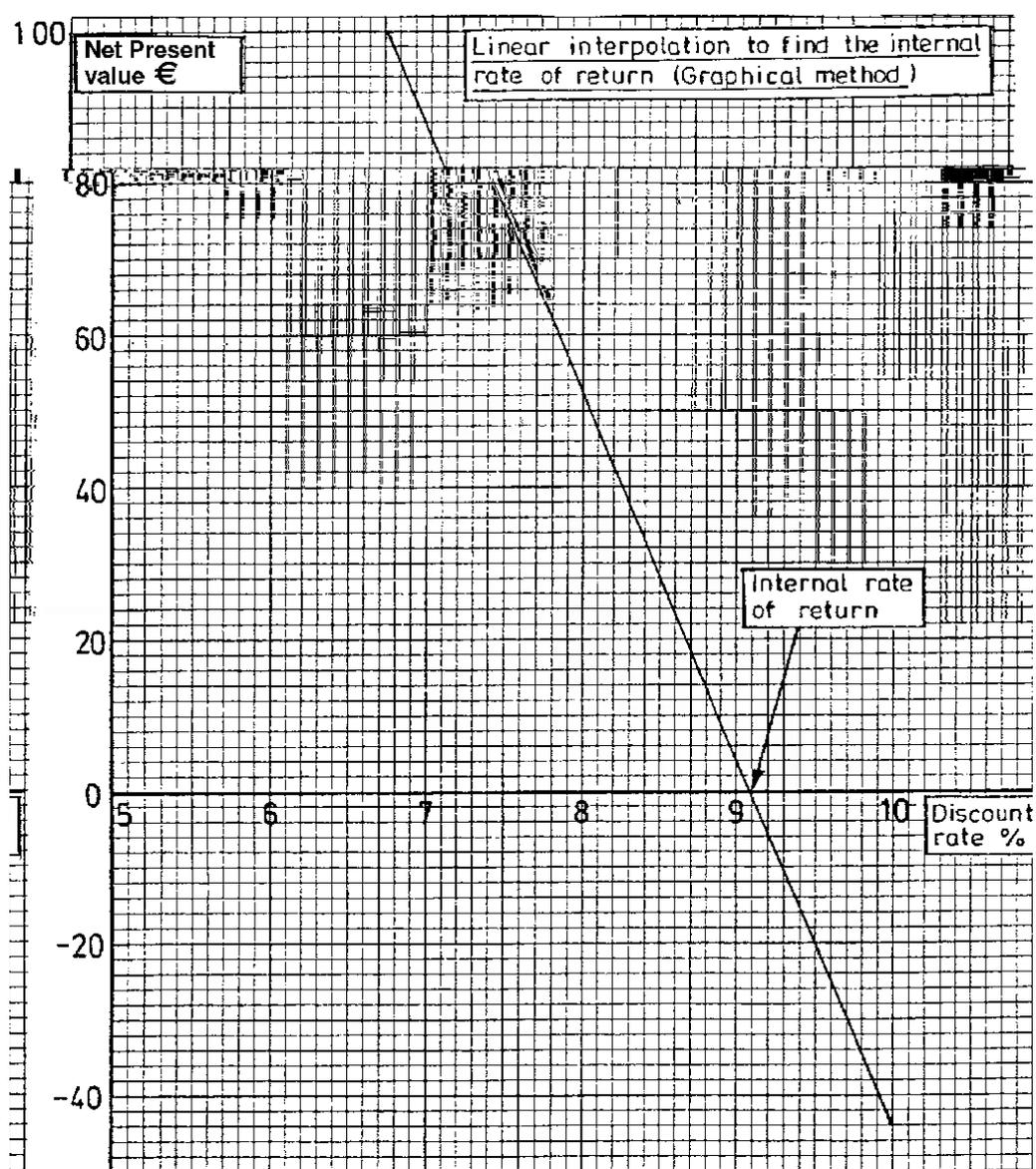


Figure 10.8

Net Present Value (NPV) Method

The NPV method is probably more widely used than the yield method, and its particular value is in comparing two or more possible investments between which a choice must be made. If a company insists on a minimum yield from investments of, say, 10%, we could check each potential project by the yield method to find out whether it measures up to this. But if there are several projects each of which yields above this figure, we still have to find some way of choosing between them if we cannot afford to undertake all of them.

At first sight the obvious choice would be that which offered the highest yield. Unfortunately this would not necessarily be the best choice, because a project with a lower yield might have a much longer life, and so might give a greater profit.

However, we can solve the problem in practice by comparing the net present values of projects instead of their yields. The higher the NPV of a project or group of projects, the greater is its value and the profits it will bring.

We must remember that in some instances the cost of capital will be higher for one project than for another. For example, a company which manufactures goods may well be able to borrow more cheaply for its normal trade than it could if it decided to take part in some more speculative process. So each project may need to be assessed at a different rate in accordance with its cost of capital. This does not present any particular problems for DCF.

NPV Method and Yield Method Contrasted

You should now be able to see the important difference between the NPV method and the yield method. In the yield method we were trying to find the yield of a project by discovering the rate at which future income must be discounted to obtain a fixed NPV of 0. In the NPV method we already know the discounting rate for each project (it will be the same as the cost of capital) and the factor which we are now trying to find for each project is its NPV. The project with the highest NPV will be the most profitable in the long run, even though its yield may be lower than other projects.

So you can see that comparison of projects by NPV may give a different result from comparison by yields. You must decide for each particular problem which method is appropriate for it.

Consider Figure 10.9, which shows the NPV profiles of two competing projects, AA and BB.

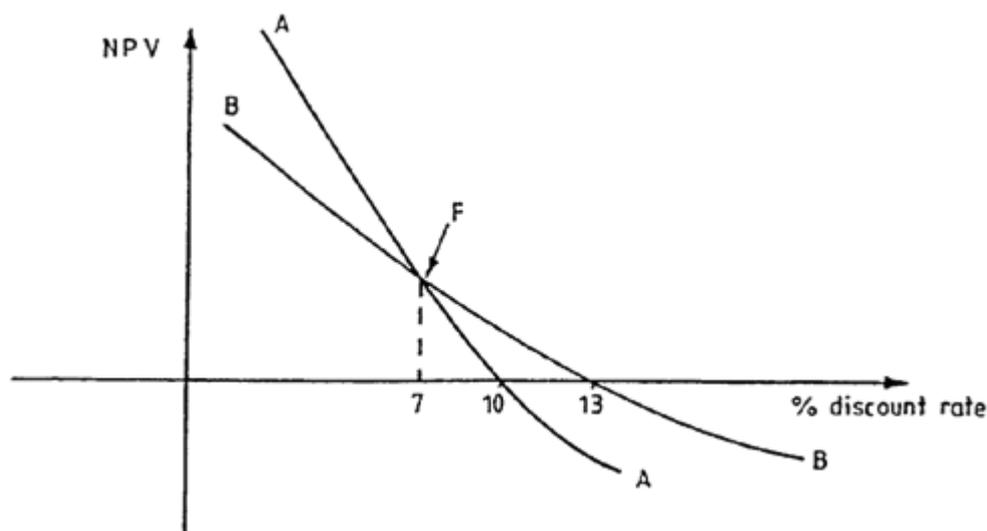


Figure 10.9

From the graph, the yield of project AA is 10% and that of project BB is 13%. The NPV of project AA is greater than that of project BB for discount rates of 0-7%, but at rates greater than 7% the NPV of project BB is greater than that of project AA.

If the company's cost of capital is 7% or less, then project AA will be preferred on an NPV basis, while project BB will be preferred on the basis of the higher yield.

If the company's cost of capital is greater than 7%, project BB will be preferred on both an NPV and yield basis.

The point F, at which the NPV profiles intersect, is called the Fisherian point, after the eminent economist, Irving Fisher. The patterns of cash flows which bring about a Fisherian point can be identified as follows:

- a) Where project life-spans vary considerably.
- b) Where the cash flows of one project begin at low levels and increase, whilst those of the other begin high and decrease. The discounting process bites more deeply into cash flows in later years because of compounding effects, whereas earlier cash flows are not so severely hit.

How to Use the NPV Method

We must first assemble the cash flow figures for each project. Then, carry out the discounting process on each annual net figure at the appropriate rate for that project, and calculate and compare the NPVs of the projects. As we have seen, that with the highest NPV will be the most profitable.

Example

The ABC Engineering Co. are trying to decide which of the two available types of machine tool to buy. Type A costs RWF10,000 and the net annual income from the first 3 years of its life will be RWF3,000, RWF4,000 and RWF5,000 respectively. At the end of this period it will be worthless except for scrap value of RWF1,000. To buy a Type A tool, the company would need to borrow from a finance group at 9%. Type B will last for three years too, but will give a constant net annual cash inflow of RWF3,000. It costs RWF6,000 but credit can be obtained from its manufacturer at 6% interest. It has no ultimate scrap value. Which investment would be the more profitable?

Table 10.10

Type A

Year	Net Cash Income	Discount Factor	Discounted Present Value
	RWF	(9%)	RWF
0	- 10,000	1.0000	- 10,000.00
1	+ 3,000	0.9174	+ 2,752.20
2	+ 4,000	0.8417	+ 3,366.80
3	+ 5,000)		
	+ 1,000)	0.7722	<u>+ 4,633.20</u>
		Net present value	<u>+ 752.20</u>

Type B

Year	Net Cash Income	Discount Factor	Discounted Present Value
	RWF	(6%)	RWF
0	- 6,000	1.0000	- 6,000.00
1	+ 3,000	0.9434	+ 2,830.20
2	+ 3,000	0.8900	+ 2,670.00
3	+ 3,000	0.8396	<u>+ 2,518.80</u>
		Net present value	<u>+ 2,019.00</u>

Thus we can see that Type B has a far higher NPV and this will be the better investment.

Allowance for Risk and Uncertainty

All investments are subject to risk. In general terms, we mean normal business risk, i.e. **not** that the investment plans will collapse completely as a total write-off, but that unforeseen factors will emerge, such as new legislation, changes in fashion, etc. which make the original estimates of costs and sales, etc. no longer valid. There are two accepted methods for incorporating risk into a capital investment appraisal:

a) Inclusion of a Risk Premium in the Discount Rate

The inclusion of a risk premium in the discount rate means that if the normal discount rate to be used were, say, 12%, then an additional amount, say 4%, might be allowed to cover for risk, making a 16% discount rate in total. The premium to be added is largely arrived at by subjective rather than objective measurement, and is correspondingly weak. As we have seen also, higher discount rates "bite" more savagely at the more distant cash flows, so that two projects, one short and one long in life-span, would be treated differently for risk by this method.

b) Attaching Probabilities to Cash Flows

With the first method we effectively looked at the project "normally" - with our usual discount rate and in a "least favourable" position, by requiring the project to provide a higher return to cover risk. We can, in fact, refine this method further by attaching individual probabilities to each cash inflow and outflow, rather than a once-off blanket cover by upping the discount rate.

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G. INTRODUCTION TO FINANCIAL MATHS

The following are the areas which we cover in this section.

Simple and compound interest, annual percentage rate (APR), depreciation (straight line and reducing balance), discounting, present value and investment appraisal, Annuities, mortgages, amortization, sinking funds.

In this area the letter i and r both stand for the interest rate. The interest rate is often referred to in financial maths as: the discount rate, the cost of capital, the rate of return.

Simple and Compound Interest

If you invest RWF100 in a bank at 10% interest then after one year it will be worth $100(1+10\%) = 100(1.1) = \text{RWF}110$The 10% is written as a decimal.

The interest here is RWF10.

If this RWF110 is left in the bank another year at 10%, then the simple interest is again RWF10 as in this case no interest is given on the previous interest earned. However, compound interest would be calculated by finding $110(1.1) = 121$.

The simple interest over 2 years is RWF20. The compound interest is RWF21.

The formula to work out the amount in your bank account after n years at $r\%$ is

$$\text{Amount } S = P (1+r)^n$$

In above example $S = 100(1.1)^2 = 121$.

Annual percentage rate (APR)

In the above example, we assumed interest was added or compounded annually, however sometimes interest may accrue ever six months (twice a year) or over 3 months (4 times per year). (This would of course be better for the customer).

If interest is at 10% per annum we call this the nominal rate, however if it is compounded every six months then the actual return is greater than 10%. We call the rate you are actually getting on your investment the effective rate or actual percentage rate (APR).

Example

the nominal rate of interest is 10% but interest is being compounded six-monthly. This means that interest is being charged at 5% per six months. Thus RWF100 invested would be worth $100(1.05)^2 = 110.25$ after 1 year so the effective rate is 10.25% and not 10%.

The APR of a nominal rate of 12% compounded quarterly = $12/4 = 3\%$ per quarter = $.03^4 = 12.55\%$.

Depreciation: Straight line and reducing balance.

Depreciation is an allowance made in estimates, valuations or balance sheets, normally for “wear and tear”. There are two techniques for calculating depreciation:

- Straight line or equal instalment depreciation &
- Reducing balance depreciation.

Straight line: if a machine is to depreciate from RWF2500 to RWF500 over 5 years then annual depreciation would be $RWF2500 - 500 = RWF2000/5 = RWF400$.

Reducing balance depreciation: remember in compounding we increased an initial investment by $(1+r)^n$, in depreciation we do a similar process in reverse.

For example, RWF2550 depreciated by 15% equals $RWF2550(1-0.15) = RWF2550(.85) = RWF2167.50$.

Also if RWF2550 was successively depreciated over four time periods by 15% the final depreciated value would be $RWF2550(.85)^4 = RWF1331.12$.

Net Present Value and Internal Rate of Return

This topic describes the technique of present value and how it can be applied to future cash flows in order to find their worth in today's money terms.

If I invest RWF100 in the bank today at 10% annual interest, then after 1 year I would have RWF110.

Looking at this in reverse, if you were due to inherit RWF110 in 1 year, and the interest rate in the bank is 10%, how much is this money worth now. In other words, how much would you need to put in the bank today in order to have RWF110 in one year? Ans: RWF100

The Present Value of RWF110 in one year's time at 10% interest is RWF100.

This is found by taking

$$\text{RWF110}/1.1 = \text{RWF110} \times 1/1.1 = \text{RWF110} \times .9090 = \text{RWF100}$$

The NPV method of investment appraisal takes into account the "time value of money". In order to assess an investment where the money earned on the investment is spread over many years the approach taken is to bring all future money amounts back to the present.

Supposing you were given the following investment options; you give me RWF10000 to invest on your behalf. I tell you that I have two different areas where I could invest your money. The return on each is given below:

Table 10.11

Year	Option 1	Option 2
1	RWF4000	RWF2000
2	RWF5000	RWF9000
3	RWF4000	RWF2500

Which option would you choose?

Table 10.12

Year	Option 1	Discount value	Present Value
1	RWF4000	.9090	RWF3636
2	RWF5000	.8264	RWF4132
3	RWF4000	.7513	RWF3005.2
The amount you would need to invest today @ 10% to have the returns indicated in column 1 is the sum of the present values			RWF10773.2

You are receiving these returns and only investing RWF10000 so your Net Present Value is

$$\text{RWF10773.20} - \text{RWF10000} = \text{RWF773.20}.$$

Since the NPV is positive, you must be receiving more than 10% on the investment.

The above problem is usually written as follows:

Table 10.13

Year	Option 1	Discount value	Present Value
0	(RWF10000)	1	(RWF10000)
1	RWF4000	.9090	RWF3636
2	RWF5000	.8264	RWF4132
3	RWF4000	.7513	RWF3005.2
NPV			+ RWF773.2

Looking at investment 1 above although with the positive NPV we know that the investment is offering a rate above the discount rate of 10%, we do not know the actual return on the investment. The Internal rate of Return gives us this figure.

What rate of return is the investment yielding?

11%, 12%, 18%??

The rate of return the investment is yielding is called the Internal Rate of Return. If I told you the internal rate of return was 16 % and you found the NPV using 16% what NPV would you expect to get?

The easiest way to find the internal rate of return is to find the NPV using two different discount rates. If the original NPV was positive use a higher rate the second time you discount.

Using option one above we already found the NPV at 10% was RWF773.2. This is positive so we will use a higher discount rate now. You can choose whatever one you want;

Let's use 20%

Table 10.14

Year	Option 1	Discount value	Present Value
0	(RWF10000)	1	(RWF10000)
1	RWF4000	.8333	3333.20
2	RWF5000	.6944	3472
3	RWF4000	.5787	2314.8
NPV			-880

This is perfect because it is a negative number which is roughly the same as the positive number done earlier.

The Internal Rate of Return is then estimated by drawing the following diagram:

$$\text{Formula: Internal Rate of Return.} \quad \frac{N_1 r_2 - N_2 r_1}{N_1 - N_2} = \frac{773.2 * 20 - (-880) * 10}{773.2 - (-880)} = 14.68$$

$$N_1 = 773.2$$

$$r_1 = 10$$

$$N_2 = -880$$

$$r_2 = 20$$

Annuities, Mortgages, Amortization, Sinking funds.

This topic deals with various techniques associated with fixed payments (or receipts) over time, otherwise known as annuities.

An annuity is a sequence of fixed equal payments (or receipts) made over uniform time intervals. Some examples are monthly salaries, insurance premiums, mortgage repayments, hire-purchase agreements.

Annuities are used in all areas of business and commerce. Loans are normally repaid with an annuity, investment funds are made up to meet fixed future commitments for example asset replacement, by the payment of an annuity. Perpetual annuities can be purchased with a single lump-sum payment to enhance pensions.

Annuities may be paid

- At the end of payment intervals (an ordinary annuity) or
- At the beginning of a payment interval (a due annuity)

There are just 2 formulae you need here:

Accrued amount (compound interest) $A = P (1+i)^n$

Sum of the first n terms of an annuity $S_n = \frac{a((1+r)^n - 1)}{r}$ This formula is used if an equal amount is lodged over many years.

Amortization of a debt.

If an amount of money is borrowed over a period of time, one way of repaying the debt is by paying an amortization annuity. This consists of a regular annuity in which each payment accounts for both repayment of capital and interest. The debt is said to be amortized if this method is used. Many of the loans issued for houses are like this. This is known as a repayment mortgage.

The standard question is: given the amount borrowed P , with interest of $r\%$, what must the annual payments be A , in order to pay off (amortize) the debt in a certain number of years.

The easiest way to do this is with an “Amortization Schedule”.

An amortization schedule is a specification, period by period (normally year by year) of the state of the debt. It is usual to show for each year:

- a) Amount of debt outstanding at the beginning of the year.
- b) Interest paid
- c) Annual payment
- d) Amount of principle repaid.

Example:

A debt of RWF5000 with interest of 5% compounded every 6 months is amortized by equal semi-annual payments over the next three years.

- a) Find the value of each payment
- b) Construct an amortization schedule.

- a) Making a standard time period of 6 months, the interest rate is 2.5% with n=6 time periods.

$$P=5000; n=6; r=0.025 \quad (1+i) = 1.025.$$

$$\text{Thus } 5000 = A \left[\frac{1}{1.025} + \frac{1}{(1.025)^2} + \frac{1}{(1.025)^3} + \frac{1}{(1.025)^4} + \frac{1}{(1.025)^5} + \frac{1}{(1.025)^6} \right]$$

$$= A (0.97561 + 0.95181 + 0.92860 + 0.90595 + 0.88385 + 0.86230)$$

$$= A (5.50812)$$

$$A = \frac{5000}{5.50812} = RWF907.75$$

- b) The amortization schedule is given below:

Table 10.15

6 month period	Outstanding debt	Interest paid	Payment made	Principal repaid
1	5000	125	907.75	782.75
2	4217.25	105.43	907.75	802.32
3	3414.93	85.37	907.75	822.38
4	2592.55	64.81	907.75	842.94
5	1749.61	43.74	907.75	864.01
6	855.6	22.14	907.75	885.61
balance	0.01			

Sinking fund

Sinking funds are commonly used for the following purposes:

- (i) Repayment of debt
- (ii) To provide funds to purchase a new asset when the existing asset is fully depreciated.

Debt repayment using a sinking fund:

Here, a debt is incurred over a fixed period of time, subject to a given interest rate. A sinking fund must be set up to mature to the outstanding amount of the debt.

For example: if RWF25000 is borrowed over 3 years at 12% compounded, the value of the outstanding debt at the end of the third year, will be $RWF25000 (1.12)^3 = RWF35123.20$.

If money can be invested at 9.5%, we need to find the value of the annuity, A, which must be paid into the fund in order that it matures to RWF35125.20. Assuming that payments into the fund are in arrears, we need:

$$35123.20 = A (1.095)^2 + A (1.095) + A$$

$$35123.2 = A (3.2940)$$

$$A = \frac{35123.2}{3.2940} = 10662.78$$

Formula Sheet

[Unless stated otherwise, all symbols have their usual meanings]

Central tendency

Arithmetic Mean

$$\bar{X} = \frac{\sum x}{n} \text{ For raw data}$$

$$\bar{X} = \frac{\sum fx}{\sum f} \text{ for grouped data}$$

Probability and Statistics

$$P(A \text{ AND } B) = P(A) \cdot P(B|A)$$

$$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B)$$

$$\text{Binomial Distribution: } P(r) = {}^n C_r p^r (1-p)^{n-r}$$

$$\text{Poisson distribution: } P(r) = \frac{e^{-\mu} \mu^r}{r!} \text{ where } \mu \text{ is the average no. of occurrences} = n \cdot p$$

$$\text{Normal Distribution: } z = \frac{x - \mu}{\sigma}$$

$$\text{Normal Approximation to the Binomial Distribution: } \sigma = \sqrt{np(1-p)} \text{ and } \mu = n \cdot p$$

$$\text{Standard Error } \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

$$\text{Confidence intervals: } \mu = \bar{x} \pm z_c \sigma_{\bar{x}} \qquad P = p \pm z_c \sqrt{\frac{p(1-p)}{n}}$$

Chi - Square formula

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

Where O is observed data
 E is expected data.

Future and present values

Future value $A = P(1 + r)^n$ where A is the amount in n years time and r is the fractional interest rate

Present value of future amount A is $\frac{A}{(1+r)^n}$;

If an amount P is invested at the beginning of a year and a further amount " a " is invested at the end of each year, then the sum, S , invested after n years is:

$$S = P(1 + r)^n + a \frac{(1 + r)^n - 1}{r}$$

Internal Rate of Return. $\frac{N_1 r_2 - N_2 r_1}{N_1 - N_2}$

Where N = Net present value, r = Discount rate.

Regression and Correlation

If the least squares regression line of y on x is given by the equation $y = a + bx$, then

$$b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

$$a = \frac{\sum y}{n} - b \frac{\sum x}{n}$$

And the product moment correlation coefficient is:

$$r = \frac{n \sum xy - \sum x \sum y}{\sqrt{n \sum x^2 - (\sum x)^2} \sqrt{n \sum y^2 - (\sum y)^2}}$$

Indices

Laspeyres Price Index

$$L = \frac{\sum p_n q_0}{\sum p_0 q_0} \times 100$$

Paasche Price Index

$$P = \frac{\sum p_n q_n}{\sum p_0 q_n} \times 100$$

Break-even Analysis

For any business there is a certain level of sales at which there is neither a profit nor a loss, i.e. the total income and the total costs are equal. This point is known as the break-even point. It is very easy to calculate, and it can also be found by drawing a graph called a break-even chart.

Calculation of Break-Even Point – Example

As shown in the last unit, you must be able to layout a marginal cost statement before doing Break Even formulas.

Marginal Cost Statement

Sales	x
- Variable Cost	(x)
= Contribution	x
- Fixed Costs	(x)
= Profit/Loss	<u>xx</u>

Let us assume that the organising committee of a dinner have set the selling price at RWF8.40 per ticket. They have agreed with a firm of caterers that the meal would be supplied at a cost of RWF5.40 per person. The other main items of expense to be considered are the costs of the premises and orchestra which will amount to RWF80 and RWF100 respectively. The variable cost in this example is the cost of catering, and the fixed costs are the amounts for premises and orchestra.

The first step in the calculations is to establish the amount of contribution per ticket.

Contribution

	RWF
Price of ticket (sales value)	8.40
Less Catering cost (marginal cost)	<u>5.40</u>
Contribution	<u>3.00</u>

Now that this has been established, we can evaluate the fixed expenses involved.

Fixed Costs

	RWF
Hire of premises	80
Orchestra fee	<u>100</u>
Total fixed expenses	RWF <u>180</u>

The organisers know that for each ticket they sell, they will obtain a contribution of RWF3 towards the fixed costs of RWF180. Clearly it is only necessary to divide RWF180 by RWF3 to establish the number of contributions which are needed to break even on the function. The break-even point is therefore 60, i.e. if 60 tickets are sold there will be neither a profit nor a loss on the function. Any tickets sold in excess of 60 will provide a profit of RWF3 each.

Formulae

The general formula for finding the **break-even** point in volume is:

$$\frac{\text{Fixed costs}}{\text{Contribution per unit}}$$

(this is, of course, exactly what we did in the example).

If the break-even point is required in terms of sales **value**, rather than sales **volume**, the formula that should be used is as follows:

$$\text{Break-even point} = \frac{\text{Fixed costs}}{\text{C/s ratio}}$$

$$\text{The C/s ratio is } \frac{\text{Contribution}}{\text{Sales}} \times 100.$$

For example, the contribution earned by selling one unit of Product A at a selling price of RWF10 is RWF4.

$$\text{C/s ratio} = \frac{\text{RWF4}}{\text{RWF10}} \times 100 = 40\%$$

In our example of the dinner-dance, the break-even point in revenue would be:

$$\frac{\text{rwf180}}{\text{rwf8.40}} = \text{RWF504}$$

The committee would know that all costs (both variable and fixed) would be exactly covered by revenue when sales revenue earned equals RWF504. At this point no profit nor loss would be received.

Suppose the committee were organising the dinner in order to raise money for charity, and they had decided in advance that the function would be cancelled unless at least RWF120 profit would be made. They would obviously want to know how many tickets they would have to sell to achieve this target.

Now, the RWF3 contribution from each ticket has to cover not only the fixed costs of RWF180, but also the desired profit of RWF120, making a total of RWF300. Clearly they will have to sell 100 tickets (RWF300 divided by RWF3).

To state this in general terms:

Volume of sales needed to achieve a given profit =

$\frac{\text{Fixed costs} + \text{Desired profit}}{\text{Contribution per unit}}$

Suppose the committee actually sold 110 tickets. Then they have sold 50 more than the number needed to break even. We say they have a **margin of safety** of 50 units, or of RWF420 (50 × RWF8.40), i.e.

$$\text{Margin of safety} = \text{Sales achieved} - \text{Sales needed to break even.}$$

The margin of safety is defined as the excess of normal or actual sales over sales at break-even point.

It may be expressed in terms of sales volume or sales revenue.

Margin of safety is very often expressed in percentage terms:

$$\frac{\text{Sales achieved} - \text{Sales needed to break even}}{\text{Sales achieved}} \times 100\%$$

i.e. the dinner committee have a percentage margin of safety of $50/110 \times 100\% = 45\%$.

The significance of margin of safety is that it indicates the amount by which sales could fall before a firm would cease to make a profit. Thus, if a firm expects to sell 2,000 units, and calculates that this would give it a margin of safety of 10%, then it will still make a profit if its sales are at least 1,800 units (2,000 – 10% of 2,000), but if its forecasts are more than 10% out, then it will make a loss.

The profit for a given level of output is given by the formula:

$$(\text{Output} \times \text{Contribution per unit}) - \text{Fixed costs.}$$

It should not, however, be necessary for you to memorise this formula, since when you have understood the basic principles of marginal costing, you should be able to work out the profit from first principles.

Consider again our example of the dinner. What would be the profit if they sold (a) 200 tickets (b) RWF840 worth of tickets?

a) We already know that the contribution per ticket is RWF3.

Therefore, if they sell 200 tickets, total contribution is $200 \times \text{RWF3} = \text{RWF600}$.

Out of this, the fixed costs of RWF180 must be covered: anything remaining is profit.

Therefore profit = RWF420. (Check: 200 tickets is 140 more than the number needed to break even. The first 60 tickets sold cover the fixed costs; the remaining 140 show a profit of RWF3 per unit. Therefore profit = $140 \times \text{RWF}3 = \text{RWF}420$, as before.)

b) RWF840 worth of tickets is 100 tickets, since they are RWF8.40 each.

	RWF
Total contribution on 100 tickets =	300
Less fixed costs	<u>180</u>
Profit	RWF120

Break-even Chart

Information Required

a) Sales Revenue

When we are drawing a break-even chart for a single product, it is a simple matter to calculate the total sales revenue which would be received at various outputs.

As an example let us take the following figures:

Output (units)	Sales revenue (RWF)
0	0
2,500	10,000
5,000	20,000
7,500	30,000
10,000	40,000

b) Fixed Costs

We must establish which elements of cost are fixed in nature. The fixed element of any semi-variable costs must also be taken into account.

Let us assume that the fixed expenses total RWF8,000.

c) Variable Costs

The variable elements of cost must be assessed at varying levels of output.

Output (units)	Variable costs (RWF)
0	0
2,500	5,000
5,000	10,000
7,500	15,000
10,000	20,000

Plotting the Graph

The graph is drawn with level of output (or sales value) represented along the horizontal axis and costs/revenues up the vertical axis. The following are the stages in the construction of the graph:

- Plot the sales line from the above figures.
- Plot the fixed expenses line. This line will be parallel to the horizontal axis.
- Plot the total expenses line. This is done by adding the fixed expenses of RWF8,000 to each of the variable costs above.
- The break-even point (often abbreviated to BEP) is represented by the meeting of the sales revenue line and the total cost line. If a vertical line is drawn from this point to meet the horizontal axis, the break-even point in terms of units of output will be found.

The graph is illustrated in Figure 10.16

Note that, although we have information available for four levels of output besides zero, one level is sufficient to draw the chart, provided we can assume that sales and costs will lie on straight lines. We can plot the single revenue point and join it to the origin (the point where there is no output and therefore no revenue). We can plot the single cost point and join it to the point where output is zero and total cost = fixed cost.

In this case, the break-even point is at 4,000 units, or a revenue of RWF16,000 (sales are at RWF4 per unit).

This can be checked by calculation:

Sales revenue	= RWF4 per unit
Variable costs	= RWF2 per unit
∴ Contribution	= RWF2 per unit
Fixed costs	= RWF8,000
Break-even point	= $\frac{\text{Fixed costs}}{\text{Contribution per unit}}$
	= 4,000 units.

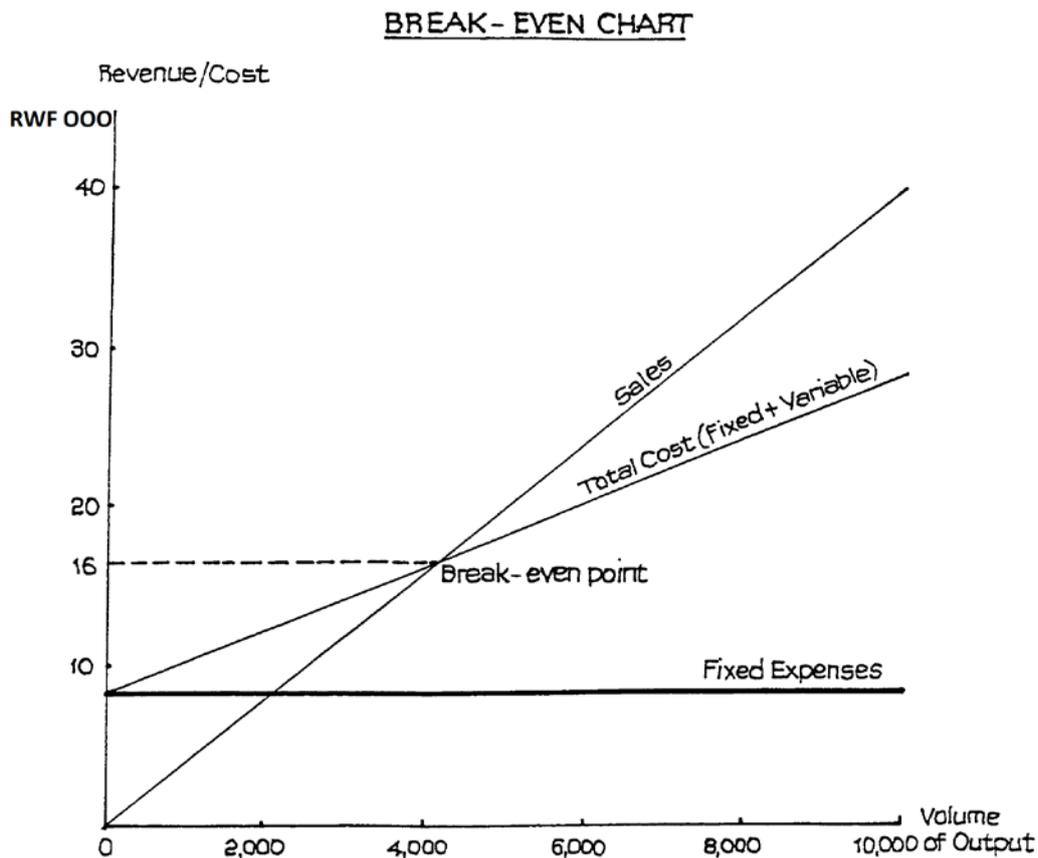


Figure 10.16

Break-even Chart for More Than One Product

Because we were looking at one product only in the above example, we were able to plot “volume of output” and straight lines were obtained for both sales revenue and costs. If we wish to take into account more than one product, it is necessary to plot “level of activity” instead of volume of output. This would be expressed as a percentage of the normal level of activity, and would take into account the mix of products at different levels of activity.

Even so, the break-even chart is not a very satisfactory form of presentation when we are concerned with more than one product: a better graph, the profit-volume graph, is discussed in the next study unit. The problem with the break-even chart is that we should find that, because of the different mixes of products at the different activity levels, the points plotted for sales revenue and variable costs would not lie on a straight line.

Fixed, Variable and Marginal Costs

Introduction

Costs can be divided either into direct and indirect costs, or variable and fixed costs.

Direct costs are **variable**, that is the total cost varies in direct proportion to output. If, for instance, it requires RWF10 worth of material to make one item it will require RWF20 worth to make two items and RWF100 worth to make ten items and so on.

Overhead costs, however, may be either fixed, variable or semi-variable.

Fixed Cost

A fixed cost is one which **can** vary with the passage of time but, **within limits**, tends to remain fixed irrespective of the variations in the level of output. All fixed costs are overhead. **Examples of fixed overhead are: executive salaries, rent, rates and depreciation.**

A graph showing the relationship of total fixed cost to output appears in Figure 10.4.

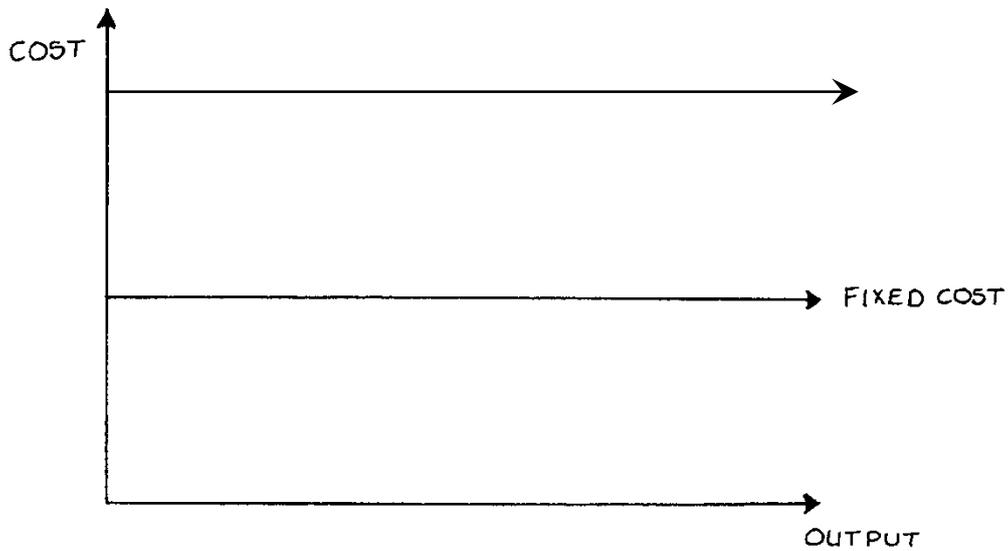


Figure 10.17

Please note the words “within limits” in the above description of fixed costs. Sometimes this is referred to as the “relevant range”, that is the range of activity level within which fixed costs (and variable costs) behave in a linear fashion.

Suppose an organisation rents a factory. The yearly rent is the same no matter what the output of the factory is. If business expands sufficiently, however, it may be that a second factory is required and a large increase in rent will follow. Fixed costs would then be as in Figure 10.5.

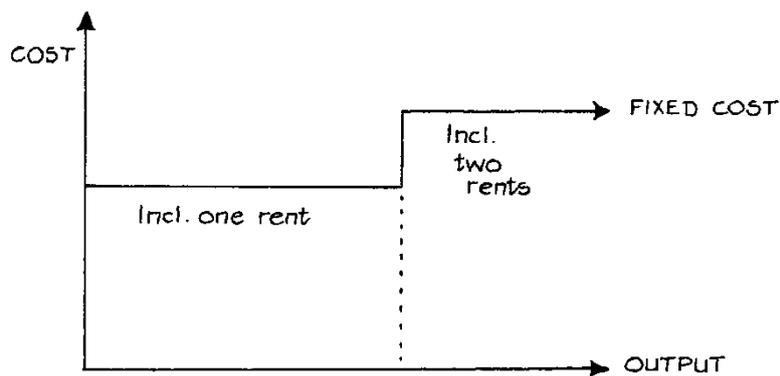


Figure 10.18

A cost with this type of graph is known as a step function cost for obvious reasons.

Variable Cost

This is a cost which tends to follow (in the short term) the level of activity in a business.

As already stated, direct costs are by their nature variable. **Examples of variable overhead are: repairs and maintenance of machinery; electric power used in the factory; consumable stores used in the factory.**

The graph of a variable cost is shown in Figure 10.6.

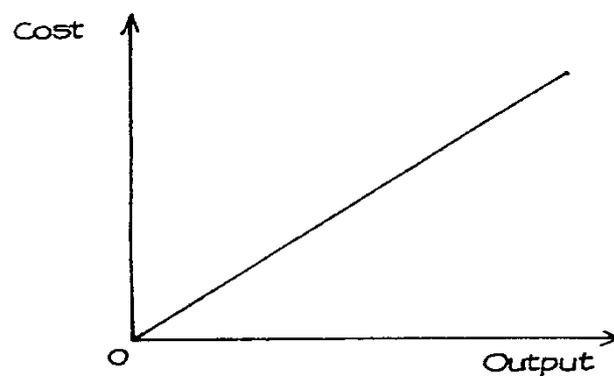


Figure 10.19

Semi-Variable (or Semi-Fixed) Cost

This is a cost containing both fixed and variable elements, and which is thus partly affected by fluctuations in the level of activity.

For examination purposes, semi-variable costs usually have to be separated into their fixed and variable components. This can be done if data is given for two different levels of output.

Example

At output 2,000 units, costs are RWF12,000.

At output 3,000 units, costs are RWF17,000.

Therefore for an extra 1,000 units of output, an extra RWF5,000 costs have been incurred. This is entirely a variable cost, so the variable component of cost is RWF5 per unit.

Therefore at the 2,000 units level, the total variable cost will be RWF10,000. Since the total cost at this level is RWF12,000, the fixed component must be RWF2,000. You can check

that a fixed component of RWF2,000 and a variable component of RWF5 per unit gives the right total cost for 3,000 units.

Step Cost

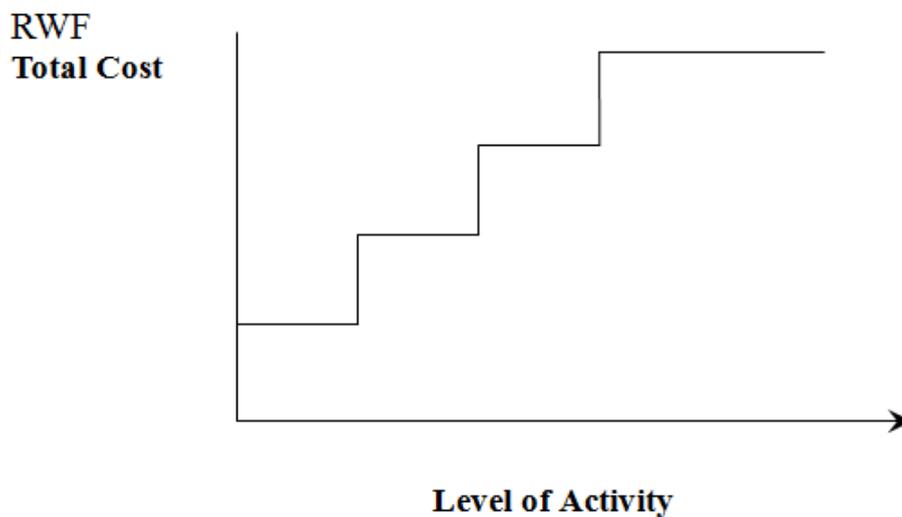


Figure 10.20

Example

Rent can be a step cost in certain situations where accommodation requirements increase as output levels get higher.

A Step Cost

Many items of cost are a fixed cost in nature within certain levels of activity.

Semi-Variable Costs

This is a cost containing both fixed and variable components and which is thus partly affected by fluctuations in the level of activity (CIMA official DFN).

Example

Running a Car

- Fixed Cost is Road Tax and insurance.
- Variable cost is petrol, repairs, oil, tyres-all of these depend on the number of miles travelled throughout the year.

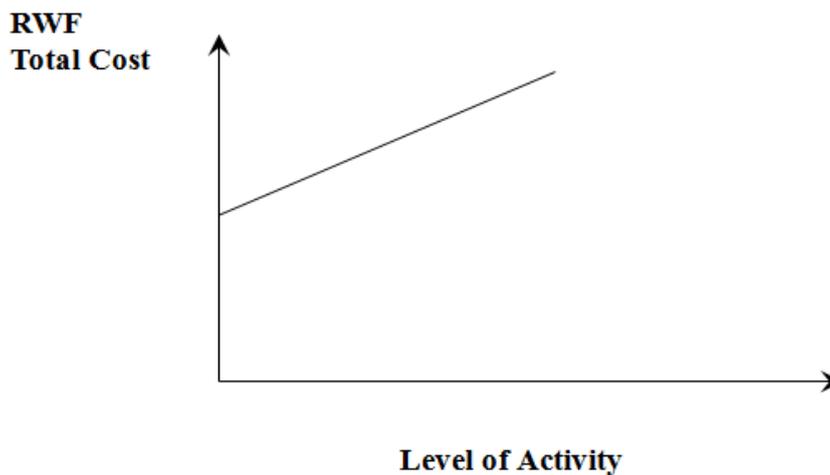


Figure 10.21

A method of splitting semi-variable costs is the High – Low method.

High – Low method

Firstly, examine records of cost from previous period. Then pick a period with the highest activity level and the period with the lowest level of activity.

- Total Cost of high activity level minus total cost of low activity level will equal variable cost of difference in activity levels.
- Fixed Costs are determined by substitution

Example of High - Low Method

Highest level 10,000 units, cost of RWF4,000

Lowest Level activity level 2,000 units cost of RWF1,600

<u>Variable Cost Element:</u>	$\frac{(\text{RWF}4,000 - \text{RWF}1,600)}{10,000 \text{ units} - 2,000 \text{ units}}$
-------------------------------	--

$$= \frac{2,400}{8,000}$$

$$\therefore = .30\text{rwf per unit}$$

Fixed Cost (under high level figure)

$$\begin{aligned} & \text{RWF4,000} - (10,000 \times .30\text{rwf}) \\ = & \text{RWF1,000} \end{aligned}$$

Scattergraphs

Information about two variables that are considered to be related in some way can be plotted on a scattergraph. This is simply a graph on which historical data can be plotted. For cost behaviour analysis, the scattergraph would be used to record cost against output level for a large number of recorded “pairs” of data.

Then by plotting cost level against activity level on a scattergraph, the shape of the resulting figure might indicate whether or not a relationship exists.

In such a scattergraph, the y axis represents cost and the x axis represents the output or activity level.

One advantage of the scattergraph is that it is possible to see quite easily if the points indicate that a relationship exists between the variables, i.e. to see if any correlation exists between them.

Positive correlation exists where the values of the variables increase together (for example, when the volume of output increases, total costs increase).

Negative correlation exists where one variable increases and the other decreases in value

Some illustrations:

1) Weight and height in humans

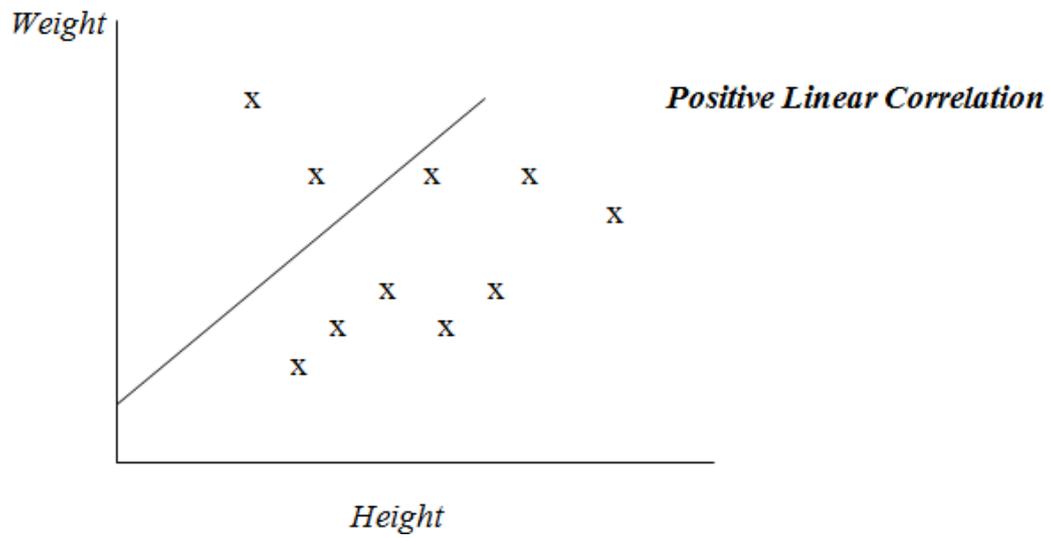


Figure 10.22

2) Sales of Scarves and temperature

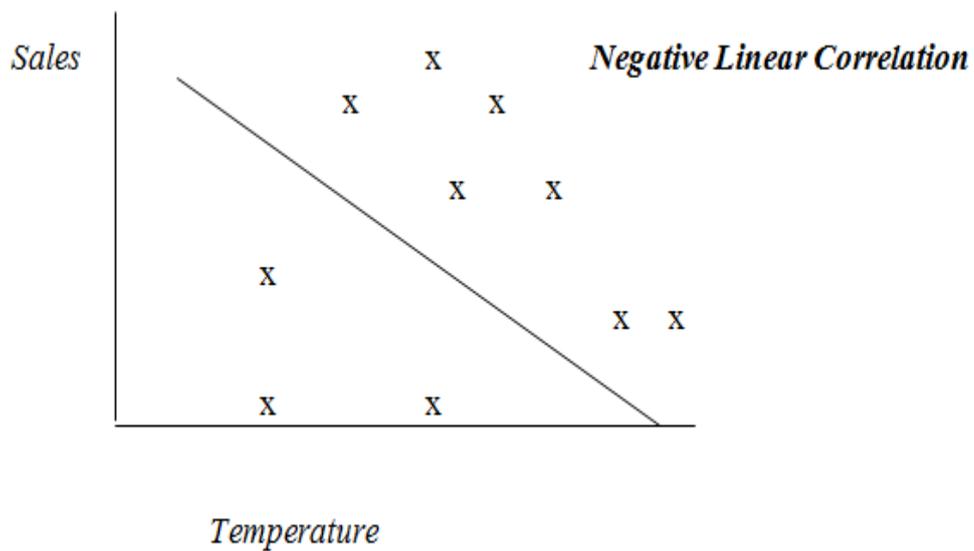


Figure 10.23

3) Sugar Imports and Mining Production

Imports

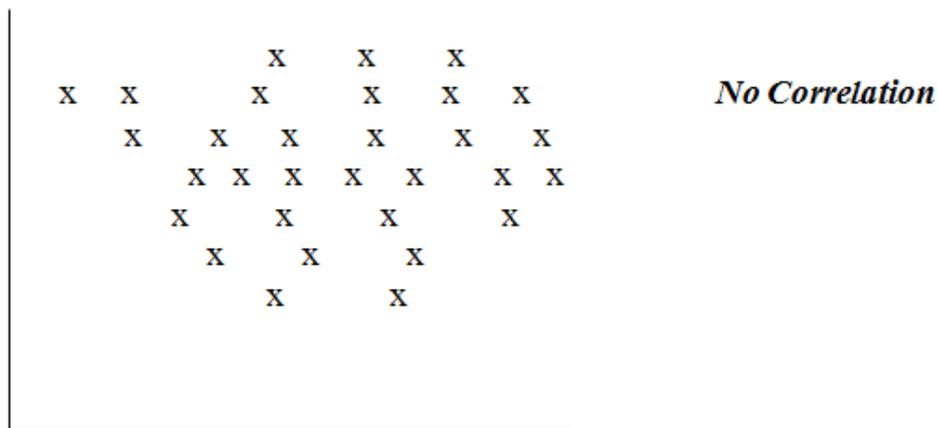


Figure 10.24

A scattergraph can be used to make an estimate of fixed and variable costs, by drawing a “*line of best fit*” through the band of points on the scattergraph, which best represents all the plotted points.

The above diagrams contain the line of best fit. These lines have been drawn using judgement. This is a major disadvantage, as drawing the line “by eye”. If there is a large amount of scatter, different people may draw different lines.

Thus, as a technique, it is only suitable where the amount of scatter is small or where the degree of accuracy of the prediction is not critical.

However, it does have an advantage over the high-low method in that all points on the graph are considered, not just the high and low point.

Regression Analysis

This is a technically superior way to identify the “slope” of the line. It is also known as “Least Squares Regression”. This statistical method is used to predict a linear relationship between two variables. It uses all past data (not just the high and low points) to calculate the line of best fit.

The equation of the regression line of y on x is of the form:

$$y = a + bx$$

In other words, if we are trying to predict the cost (y) from an activity (x), it is necessary to calculate the values of a and b from given pairs of data for x and y. The following formulae are used:

$$a = \frac{\sum y - b \sum x}{n}$$
$$b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

where “n” is the number of pairs of x and y values. (The symbol “Σ” means ‘the sum of’)

Thus, in order to calculate “a”, it is necessary to calculate “b” first.

Example

The following is the output of a factory and the cost of production over the last 5 months:

Table 10.25

	Output ('000 units)	Cost (RWF'000)
January	20	82
February	16	70
March	24	90
April	22	85
May	18	73

- (i) Determine a formula to show the expected level of costs for any given volume of output
- (ii) Prepare a budget for total costs if output is 27,000 units

Solution:

Let x = output

Let y = costs

n = 5 (5 pairs of x & y values)

Construct a table as follows: (in '000)

Table 10.26

x	y	xy	x ²	y ²
20	82	1,640	400	6,724
16	70	1,120	256	4,900
24	90	2,160	576	8,100
22	85	1,870	484	7,225
18	73	1,314	324	5,329
Σx = 100	Σy = 400	Σxy = 8,104	Σx² = 2,040	Σy² = 32,278

$$b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} = \frac{5(8,104) - (100)(400)}{5(2,040) - (100)^2}$$

$$b = 2.60$$

$$a = \frac{\sum y - b \sum x}{n} = \frac{400 - 2.6(100)}{5}$$

$$a = 28 \text{ (or 28,000)}$$

Thus, the formula for any given level of output is:

$$y = \text{RWF}28,000 + \text{RWF}2.60x$$

where

$$y = \text{total cost (in RWF'000)}$$

$$x = \text{output (in '000 units)}$$

If output is 27,000 units, then total cost (y) will be:

$$y = \text{RWF}28,000 + \text{RWF}2.60(27,000)$$

$$y = \text{RWF}98,200$$

The Linear Assumption of Cost Behaviour

1. Cost are assumed to be either fixed, variable or semi-variable within a normal range of output.
2. Fixed and variable costs can be estimated with degrees of probable accuracy. Certain methods maybe used to access this (High-Low method).
3. Costs will rise in a straight line/linear fashion as the activity increases.

Accountants – V's Economist Model

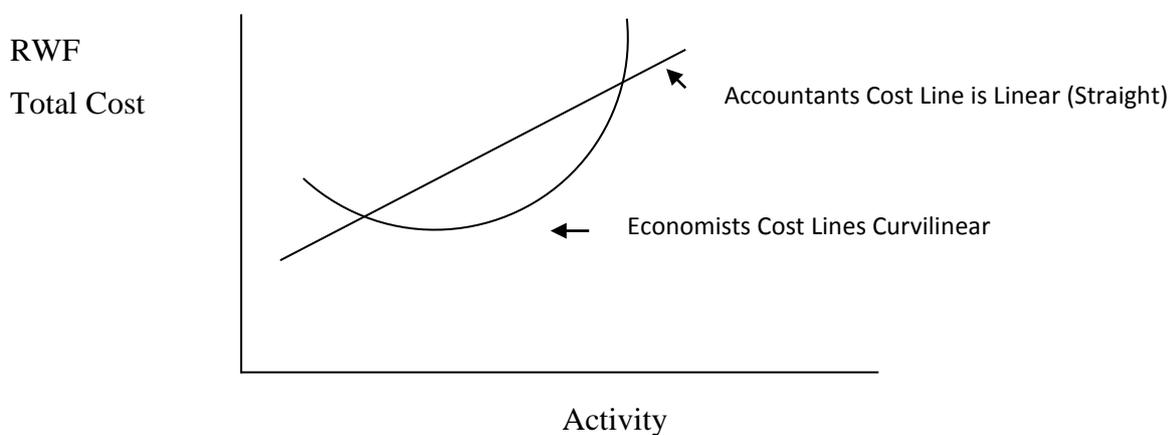


Figure 10.27

Assumptions of Above Diagram

The accountants state that the linear assumption of cost behaviour is linear because:

- 1) The linear cost (straight line) is only used in practice within normal ranges of output 'Relevant Range of Activity'.

The term 'Relevant Range' is used to refer to the output range at which the firm expects to be operating in the future.

- 2) It is easier to understand than Economists' cost line.
- 3) The fixed and variable costs are easier to use.
- 4) The relevant range and the costs estimated by the economists and the accountants will not be very different.

Factors Affecting the Activity Level

- 1) The economic environment.
- 2) The individual firm – its staff, their motivation and industrial relations.
- 3) The ability and talent of management.
- 4) The workforce (unskilled, semi-skilled and highly skilled).
- 5) The capacity of machines.
- 6) The availability of raw material.

Cost Behaviour and Decision Making

Factors to Consider:

- 1) Future plans for the company.
- 2) Current competition to the company.
- 3) Should the selling price of a single unit be reduced in order to attract more customers.
- 4) Should sale staff be on a fixed salary or on a basic wage with bonus/commission.
- 5) Is a new machine required for current year.
- 6) Will the company make the product internally or buy it.

For all of the above factors, management must estimate costs at all levels and evaluate different courses of action. Management must take all eventualities into account when making decisions for the company.

Example of things management would need to know is fixed costs do not generally change as a result of a decision unless the company have to rent an additional building for a new job etc.

Cost Variability and Inflation

Care must be taken in interpreting cost data over a period of time if there is inflation. It may appear that costs have risen relative to output, but this may be purely because of inflation rather than because the amount of resources used has increased.

If a cost index, such as the Retail Price Index, is available the effects of inflation can be eliminated and the true cost behaviour pattern revealed.

It is essential for the index selected to be relevant to the company; if one of the many Central Statistical Office indices is not appropriate, it may be possible for the company to construct one from its own data.

Consider the following example, which deals with the relationship between production output and the total costs of a single-product company, taken over a period of four years:

Year	Output <i>(tonnes)</i>	Total Costs <i>RWF</i>
1	2,700	10,400
2	3,100	11,760
3	3,700	14,880
4	4,400	20,700

Suppose that we have the above information, together with the cost indices as follows:

Year	Cost Index
1	100
2	105
3	120
4	150
5	175 (estimated)

If our estimated output for Year 5 is 5,000 tonnes, how may we calculate the estimated total costs?

First, we have to convert the costs of the four years' production to Year 1 cost levels, by applying the indices as follows:

Year	Actual Cost	Conversion Factor	Cost at Year 1 Level
	RWF		RWF
1	10,400	1	10,400
2	11,760	100/105	11,200
3	14,880	100/120	12,400
4	20,700	100/150	13,800

Secondly, we must split the adjusted costs into their fixed and variable elements. This is done by examining the difference or movement between any two years, for example:

	Production	Adjusted Cost
Year 1	2,700 tonnes	RWF10,400
Year 4	4,400 tonnes	RWF13,800

We observe that an increase of 1,700 tonnes gives a rise in costs of RWF3,400. The variable cost is therefore RWF2 per tonne.

Now by deducting the variable cost from the adjusted cost in **any** year, we can ascertain the level of fixed cost. For example, in Year 4, the variable cost @ RWF2 per tonne would be $4,400 \times \text{RWF}2 = \text{RWF}8,800$. If we deduct this figure from the total adjusted cost RWF13,800, we are left with the fixed cost total of $\text{RWF}13,800 - \text{RWF}8,800 = \text{RWF}5,000$. This fixed cost is, of course, expressed in terms of Year 1 cost level. In real terms, the fixed costs (those costs which do not vary with changes in volume) will increase over the four years in proportion to the cost index.

We now see that the yearly total costs, adjusted to Year 1 cost levels, may be split into the fixed and variable elements as follows:

Year	Production (tonnes)	Fixed RWF	Variable @ RWF2 tonne	Total RWF
1	2,700	5,000	5,400	10,400
2	3,100	5,000	6,200	11,200
3	3,700	5,000	7,400	12,400
4	4,400	5,000	8,800	13,800
5 (est'd)	5,000	5,000	10,000	15,000

Finally, by applying the cost index for each year to the total costs at Year 1 cost levels, we may complete our forecast:

Year RWF	Total Cost at Year 1 Levels RWF	Cost Index	Actual Cost
1	10,400	100	10,400
2	11,200	105	11,760
3	12,400	120	14,880
4	13,800	150	20,700
5 (est'd)	15,000	175	26,250

Limitations

This forecast of RWF26,250 for the total costs in Year 5 is, of course, subject to many limitations. The method of calculation assumes that all costs are either absolutely fixed or are variable in direct proportion to the volume of production. In practice, as we have seen, it is usually found that “fixed” costs will tend to rise slightly in steps, while the variable costs will usually rise less steeply at the higher levels of output, because of the economies of scale.

Also, our forecast will only be as accurate as our forecast of the cost index for Year 5. This is as difficult to predict as the Retail Price Index, which is influenced by changes in the price of each item in the “shopping basket”.

The analysis of cost behaviour in this way is thus useful as a guide to management, provided we remember that:

- a) It assumes a linear (or “straight line”) relationship between volume and cost.
- b) Costs will be influenced by many other factors, such as new production methods or new plant.
- c) Inflation will have a varying effect on different items of cost.

This subject of cost behaviour is fundamental to many aspects of cost accounting.

Calculus

Introduction

Calculus is the study of change. It constitutes a major part of modern mathematics education. It has **two** major branches:-

- 1) Differential Calculus
- 2) Integral Calculus

Differential calculus

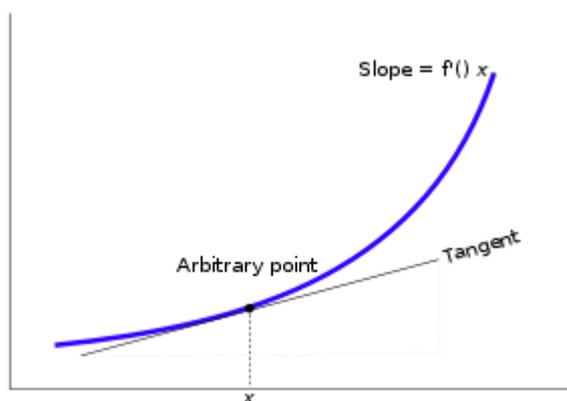


Figure 10.28

Tangent line at $(x, f(x))$. The derivative $f'(x)$ of a curve at a point is the slope (rise over run) of the line tangent to that curve at that point.

Differential calculus is the study of the definition, properties, and applications of the **derivative** of a function. The process of finding the derivative is called **differentiation**. Given a function and a point in the domain, the derivative at that point is a way of encoding the small-scale behavior of the function near that point. By finding the derivative of a function at every point in its domain, it is possible to produce a new function, called the **derivative function** or just the **derivative** of the original function. In mathematical terms the derivative is a **linear operator** which inputs a function and outputs a second function. This is more abstract than many of the processes studied in elementary algebra, where functions usually input a number and output another number. For example, if the doubling function is given the input three, then it outputs six, and if the squaring function is given the input three, then it outputs nine. The derivative, however, can take the squaring function as an input. This means that the derivative takes all the information of the squaring function—such as that two is sent to four, three is sent to nine, four is sent to sixteen, and so on—and uses this information to produce another function. (The function it produces turns out to be the doubling function.)

The most common symbol for a derivative is an apostrophe-like mark called **prime**. Thus, the derivative of the function of f is f' , pronounced "f prime." For instance, if $f(x) = x^2$ is the squaring function, then $f'(x) = 2x$ is its derivative, the doubling function.

If the input of the function represents time, then the derivative represents change with respect to time. For example, if f is a function that takes a time as input and gives the position of a ball at that time as output, then the derivative of f is how the position is changing in time, that is, it is the **velocity** of the ball.

If a function is **linear** (that is, if the **graph** of the function is a straight line), then the function can be written as $y = mx + b$, where x is the independent variable, y is the dependent variable, b is the y -intercept, and:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

This gives an exact value for the slope of a straight line. If the graph of the function is not a straight line, however, then the change in y divided by the change in x varies. Derivatives give an exact meaning to the notion of change in output with respect to change in input. To be concrete, let f be a function, and fix a point a in the domain of f . $(a, f(a))$ is a point on the graph of the function. If h is a number close to zero, then $a + h$ is a number close to a . Therefore $(a + h, f(a + h))$ is close to $(a, f(a))$. The slope between these two points is

$$m = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}.$$

This expression is called a **difference quotient**. A line through two points on a curve is called a **secant line**, so m is the slope of the secant line between $(a, f(a))$ and $(a + h, f(a + h))$. The secant line is only an approximation to the behavior of the function at the point a because it does not account for what happens between a and $a + h$. It is not possible to discover the behavior at a by setting h to zero because this would require dividing by zero, which is impossible. The derivative is defined by taking the **limit** as h tends to zero, meaning that it considers the behavior of f for all small values of h and extracts a consistent value for the case when h equals zero:

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

Geometrically, the derivative is the slope of the tangent line to the graph of f at a . The tangent line is a limit of secant lines just as the derivative is a limit of difference quotients. For this reason, the derivative is sometimes called the slope of the function f .

Here is a particular example, the derivative of the squaring function at the input 3. Let $f(x) = x^2$ be the squaring function.

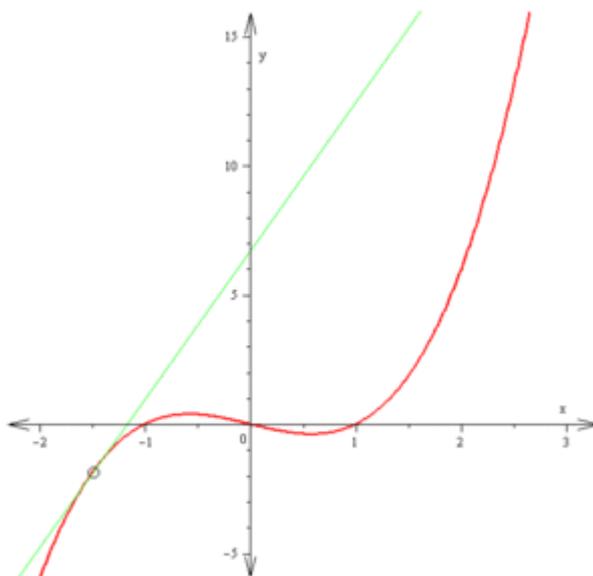


Figure 10.29

The derivative $f'(x)$ of a curve at a point is the slope of the line tangent to that curve at that point. This slope is determined by considering the limiting value of the slopes of secant lines. Here the function involved is $f(x) = x^3 - x$. The tangent line which passes through the point $(-3/2, -15/8)$ has a slope of $23/4$. Note that the vertical and horizontal scales in this image are different.

$$\begin{aligned}
 f'(3) &= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} (6 + h) \\
 &= 6.
 \end{aligned}$$

The slope of tangent line to the squaring function at the point (3,9) is 6, that is to say, it is going up six times as fast as it is going to the right. The limit process just described can be performed for any point in the domain of the squaring function. This defines the **derivative function** of the squaring function, or just the **derivative** of the squaring function for short. A similar computation to the one above shows that the derivative of the squaring function is the doubling function.

Integral calculus

Integral calculus is the study of the definitions, properties, and applications of two related concepts, the *indefinite integral* and the *definite integral*. The process of finding the value of an integral is called *integration*. Integral calculus studies two related **linear operators**.

The **indefinite integral** is the **antiderivative**, the inverse operation to the derivative. F is an indefinite integral of f when f is a derivative of F . (This use of lower- and upper-case letters for a function and its indefinite integral is common in calculus.)

The *definite integral* inputs a function and outputs a number, which gives the algebraic sum of areas between the graph of the input and the **x-axis**. The technical definition of the definite integral is the **limit** of a sum of areas of rectangles, called a **Riemann sum**.

A motivating example is the distances travelled in a given time.

$$\text{Distance} = \text{Speed} \times \text{Time}$$

If the speed is constant, only multiplication is needed, but if the speed changes, then we need a more powerful method of finding the distance. One such method is to approximate the distance travelled by breaking up the time into many short intervals of time, then multiplying the time elapsed in each interval by one of the speeds in that interval, and then taking the sum (a **Riemann sum**) of the approximate distance travelled in each interval. The basic idea is that if only a short time elapses, then the speed will stay more or less the same. However, a Riemann sum only gives an approximation of the distance traveled. We must take the limit of all such Riemann sums to find the exact distance traveled.

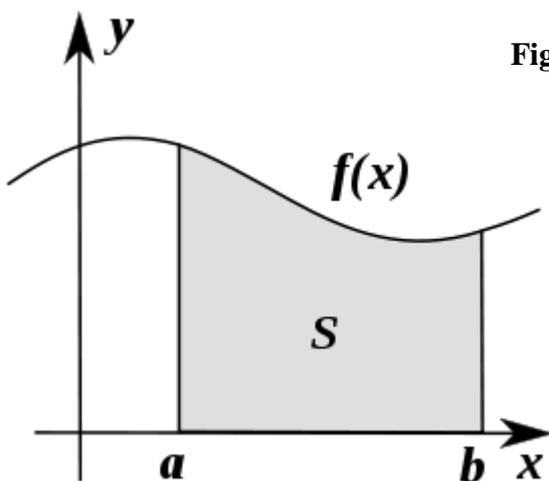


Figure 10.30

Integration can be thought of as measuring the area under a curve, defined by $f(x)$, between two points (here a and b).

If $f(x)$ in the diagram on the left represents speed as it varies over time, the distance traveled (between the times represented by a and b) is the area of the shaded region s .

To approximate that area, an intuitive method would be to divide up the distance between a and b into a number of equal segments, the length of each segment represented by the symbol Δx . For each small segment, we can choose one value of the function $f(x)$. Call that value h . Then the area of the rectangle with base Δx and height h gives the distance (time Δx multiplied by speed h) traveled in that segment. Associated with each segment is the average value of the function above it, $f(x)=h$. The sum of all such rectangles gives an approximation of the area between the axis and the curve, which is an approximation of the total distance traveled. A smaller value for Δx will give more rectangles and in most cases a better approximation, but for an exact answer we need to take a limit as Δx approaches zero.

The symbol of integration is , an elongated S (the S stands for "sum"). The definite integral is written as:

$$\int_a^b f(x) dx.$$

and is read "the integral from a to b of f -of- x with respect to x ." The Leibniz notation dx is intended to suggest dividing the area under the curve into an infinite number of rectangles, so that their width Δx becomes the infinitesimally small dx . In a formulation of the calculus based on limits, the notation

$$\int_a^b \dots dx$$

is to be understood as an operator that takes a function as an input and gives a number, the area, as an output; dx is not a number, and is not being multiplied by $f(x)$.

The indefinite integral, or antiderivative, is written:

$$\int f(x) dx.$$

Functions differing by only a constant have the same derivative, and therefore the antiderivative of a given function is actually a family of functions differing only by a constant. Since the derivative of the function $y = x^2 + C$, where C is any constant, is $y' = 2x$, the antiderivative of the latter is given by:

$$\int 2x \, dx = x^2 + C.$$

An undetermined constant like C in the antiderivative is known as a **constant of integration**.

Fundamental theorem

The **fundamental theorem of calculus** states that differentiation and integration are inverse operations. More precisely, it relates the values of antiderivatives to definite integrals. Because it is usually easier to compute an antiderivative than to apply the definition of a definite integral, the Fundamental Theorem of Calculus provides a practical way of computing definite integrals. It can also be interpreted as a precise statement of the fact that differentiation is the inverse of integration.

The Fundamental Theorem of Calculus states: If a function f is continuous on the interval $[a, b]$ and if F is a function whose derivative is f on the interval (a, b) , then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Furthermore, for every x in the interval (a, b) ,

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

STUDY UNIT 11

Correlation

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A. GENERAL

When studying frequency distributions, we were always handling only **one variable**, e.g. height or weight. Having learned how to solve problems involving only one variable, we should now discover how to solve problems involving **two variables** at the same time.

If we are comparing the weekly takings of two or more firms, we are dealing with only one variable, that of takings; if we are comparing the weekly profits of two or more firms, we are dealing with only one variable, that of profits. But if we are trying to assess, for one firm (or a group of firms), whether there is any relationship between takings and profits, then we are dealing with two variables, i.e. takings and profits.

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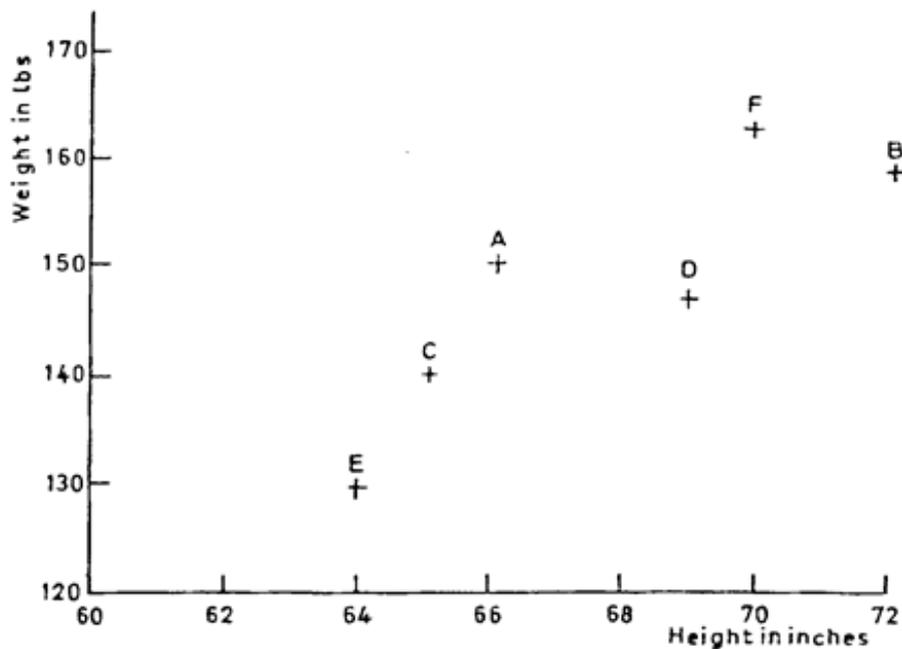
B. SCATTER DIAGRAMS

Examples of Correlation

Table 11.1

Man	Height (ins)	Weight (lb)
A	66	150
B	72	159
C	65	138
D	69	145
E	64	128
F	70	165

A **scatter diagram** or scattergram is the name given to the method of representing these figures graphically. On the diagram, the horizontal scale represents one of the variables (let's say height) while the other (vertical) scale represents the other variable (weight). Each **pair** of measurements is represented by one point on the diagram, as shown in Figure 11.1:



Scattergram of Men's Heights and Weights

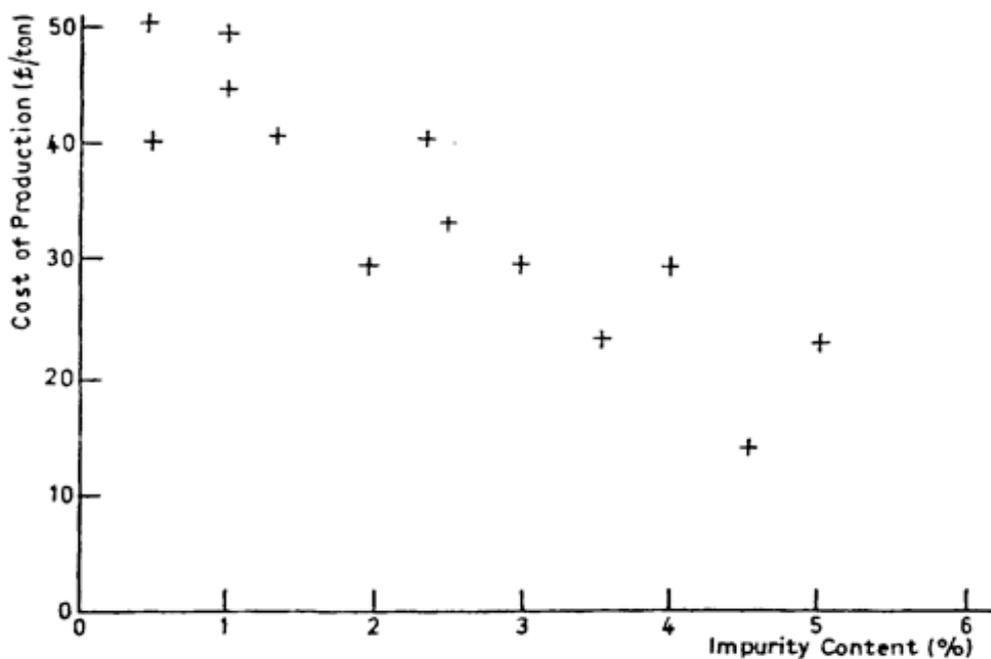
Figure 11.1

Make sure that you understand how to plot the points on a scatter diagram, noting especially that:

- Each point represents a PAIR of corresponding values.
- The two scales relate to the two variables under discussion.

The term scatter diagram or scattergram comes from the scattered appearance of the points on the chart.

Examining the scatter diagram of heights and weights, you can see that it shows up the fact that, by and large, tall men are heavier than short men. This shows that some relationship exists between men's heights and weights. We express this in statistical terms by saying that the two variables, height and weight are CORRELATED. Figure 11.2 shows another example of a pair of correlated variables (each point represents one production batch):



Cost of Production Compared with Impurity Contents

Figure 11.2

Here you see that, in general, it costs more to produce material with a low impurity content than it does to produce material with a high impurity content. However, you should note that correlation does not necessarily mean an exact relationship, for we know that, while tall men are usually heavy, there are exceptions, and it is most unlikely that several men of the same height will have exactly the same weight!

Degrees of Correlation

In order to generalise our discussion, and to avoid having to refer to particular examples such as height and weight or impurity and cost, we will refer to our two variables as x and y . On scatter diagrams, the horizontal scale is always the x scale and the vertical scale is always the y scale. There are three degrees of correlation which may be observed on a scatter diagram.

The two variables may be:

a) Perfectly Correlated

When the points on the diagram all lie exactly on a straight line (Figure 11.3):

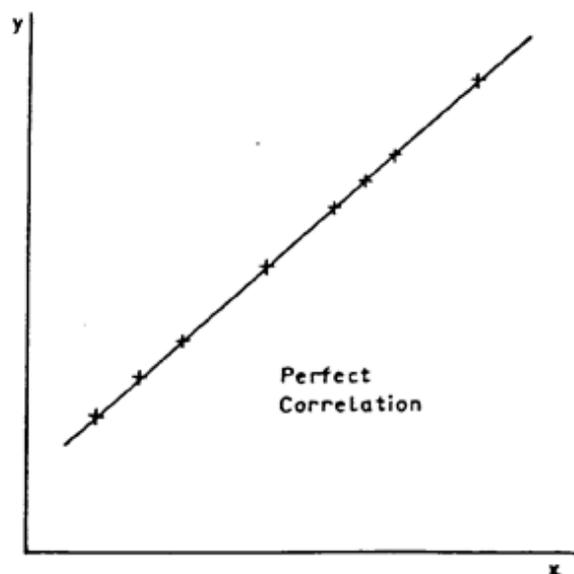


Figure 11.3

b) Uncorrelated

When the points on the diagram appear to be randomly scattered about, with no suggestion of any relationship (Figure 11.4):

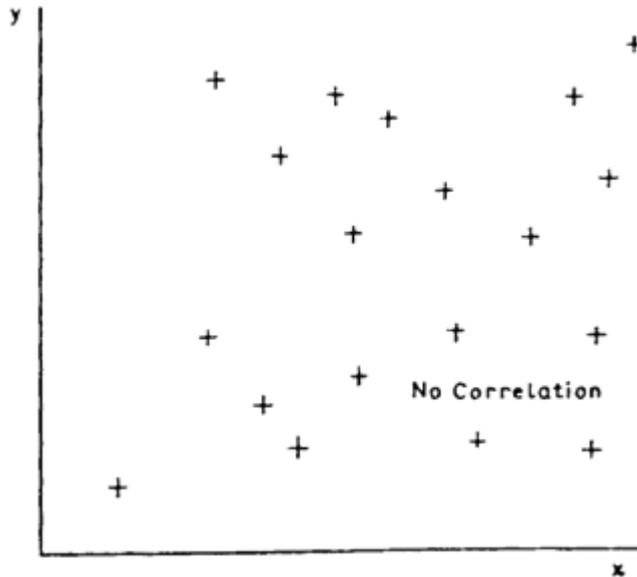
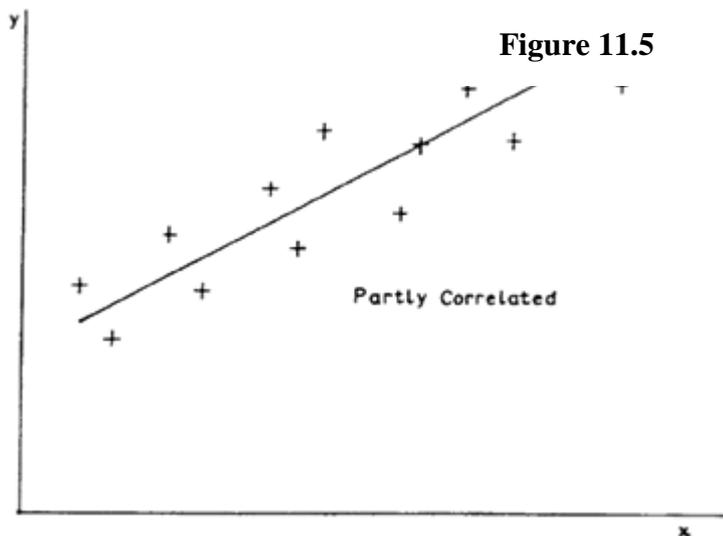


Figure 11.4

c) Partly Correlated

When the points lie scattered in such a way that, although they do not lie exactly on a straight line, they do display a general tendency to be clustered around such a line (Figure 11.5):



Different Types of Correlation

There is a further distinction between correlations of the height/weight type and those of the impurity/cost type. In the first case, high values of the x variable are associated with high values of the y variable, while low values of x are associated with low values of y. On the scatter diagram (Figure 11.6 (a)), the points have the appearance of clustering about a line which slopes **up to the right**. Such correlation is called **POSITIVE** or **DIRECT** correlation.

In the other case (like the impurity/cost relationship) high values of the x variable are associated with low values of the y variable and vice versa; on the scatter diagram (Figure 11.6 (b)) the approximate line slopes **down to the right**. This correlation is said to be **NEGATIVE** or **INVERSE**.

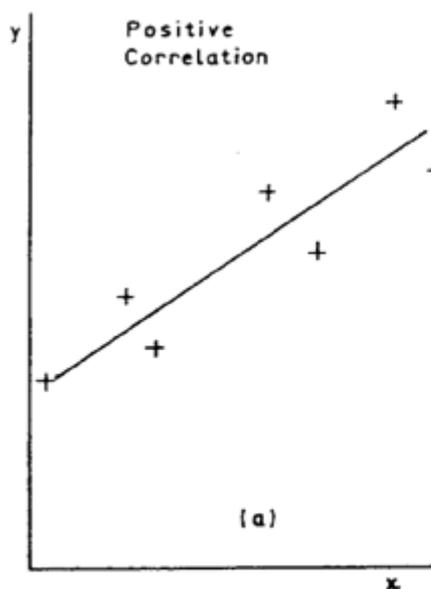


Figure 11.6 (a)

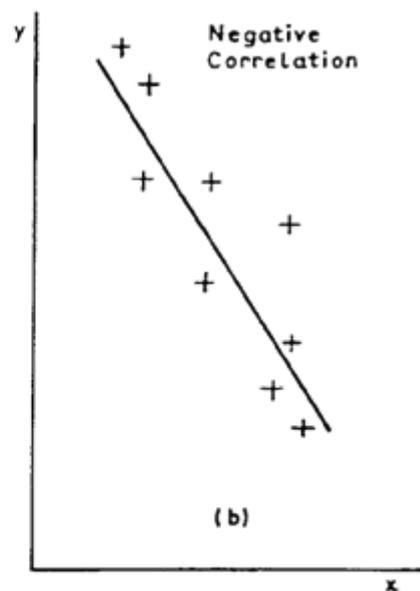


Figure 11.6 (b)

a) Linear Correlation

The correlation is said to be linear when the relationship between the two variables is linear. In other words all the points can be represented by straight lines. For example, the correlation

between car ownership and family income may be linear as car ownership is related in a linear fashion to family income.

b) Non-linear Correlation

Non-linear correlation is outside the scope of this course but it is possible that you could be required to define it in an examination question. It occurs when the relationship between the two variables is non-linear. An example is the correlation between the yield of a crop, like carrots, and rainfall. As rainfall increases so does the yield of the crop of carrots, but if rainfall is too large the crop will rot and yield will fall. Therefore, the relationship between carrot production and rainfall is non-linear.

C. THE CORRELATION COEFFICIENT

General

If the points on a scatter diagram all lie very close to a straight line, then the correlation between the two variables is stronger than it is if the points lie fairly widely scattered away from the line.

To measure the strength, or intensity, of the correlation in a particular case, we calculate a **LINEAR CORRELATION COEFFICIENT**, which we indicate by the small letter r . In textbooks and examination papers you will sometimes find this referred to as Pearson's Product Moment Coefficient of Linear Correlation, after the English statistician who invented it. It is also known as the product-moment correlation coefficient.

For an illustration of the method used to calculate the correlation coefficient, suppose we are given the following pairs of values of x and y :

x	10	14	7	12	5	6
y	5	3	5	2	7	8

Table 11.2

We shall plot these on a scatter diagram so that we can make some qualitative assessment of the type of correlation present (Figure 11.7). We see from the scatter diagram that some negative correlation

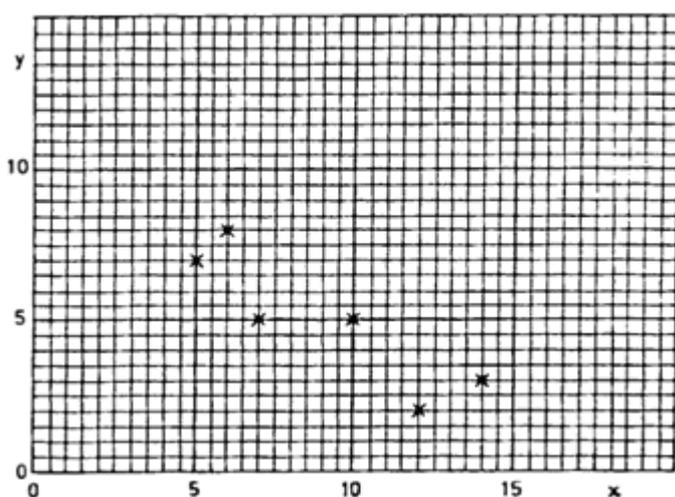


Figure 11.7

Scatter Diagram

Formula

The formula for Pearson's product-moment correlation coefficient is:

$$r = \frac{n \sum xy - \sum x \sum y}{\sqrt{(n \sum x^2 - (\sum x)^2)(n \sum y^2 - (\sum y)^2)}}$$

n is the number of pairs of readings. It is a good idea to set out the calculation in tabular form.

Table 11.3

x	y	x ²	y ²	xy
10	5	100	25	50
14	3	196	9	42
7	5	49	25	35
12	2	144	4	24
5	7	25	49	35
6	8	36	64	48
$\Sigma x = 54$	$\Sigma y = 30$	$\Sigma x^2 = 550$	$\Sigma y^2 = 176$	$\Sigma xy = 234$

$$n = 6$$

$$r = \frac{6 \times 234 - 54 \times 30}{\sqrt{(6 \times 550 - 54^2)(6 \times 176 - 30^2)}}$$

$$r = \frac{1,404 - 1,620}{\sqrt{(3,300 - 2,916)(1,056 - 900)}}$$

$$= \frac{-216}{\sqrt{384 \times 156}} = \frac{-216}{\sqrt{59,904}} = \frac{-216}{244.75} = 0.88 \text{ to 2 decimal places.}$$

This result ($r = -0.88$) shows that x and y are negatively correlated.

Characteristics of a Correlation Coefficient

We know what the + and - signs of the correlation coefficient tell us: that the relationship is positive (increase of x goes with increase of y) or negative (increase of x goes with decrease of y). But what does the actual numerical value mean? Note the following points:

- a) The correlation coefficient is always between -1 and +1 inclusive. If you get a numerical value bigger than 1, then you've made a mistake!
- b) A correlation coefficient of -1.0 occurs when there is **PERFECT NEGATIVE CORRELATION**, i.e. all the points lie **EXACTLY** on a straight line sloping down from left to right.
- c) A correlation of 0 occurs when there is **NO CORRELATION**.
- d) A correlation of +1.0 occurs when there is **PERFECT POSITIVE CORRELATION**, i.e. all the points lie **EXACTLY** on a straight line sloping upwards from left to right.
- e) A correlation of between 0 and ± 1.0 indicates that the variables are **PARTLY CORRELATED**. This means that there is a relationship between the variables but that the results have also been affected by other factors.

In our example ($r = -0.88$), we see that the two variables are quite strongly negatively correlated. If the values of r had been, say, -0.224, we should have said that the variables were only slightly negatively correlated. For the time being, this kind of interpretation is all that you need consider.

Significance of the Correlation Coefficient

Correlation analysis has been applied to data from many business fields and has often proved to be extremely useful. For example, it has helped to locate the rich oil fields in the North Sea and also helps the stockbroker to select the best shares in which to put his clients' money.

Like many other areas of statistical analysis, correlation analysis is usually applied to sample data. Thus the coefficient, like other statistics derived from samples, must be examined to see how far they can be used to make generalised statements about the population from which the samples were drawn. **Significance** tests for the correlation coefficient are possible to make, but they are beyond the scope of this course, although you should be aware that they exist.

We must be wary of accepting a high correlation coefficient without studying what it means. Just because the correlation coefficient says there is some form of association, we should not accept it without some other supporting evidence. We must also be wary of drawing conclusions from data that does not contain many pairs of observations. Since the sample size is used to calculate the coefficient, it will influence the result and, whilst there are no hard and fast rules to apply, it may well be that a correlation of 0.8 from 30 pairs of observations is a more reliable statistic than 0.9 from 6 pairs.

Another useful statistic is r^2 (r squared); this is called the **coefficient of discrimination** and may be regarded as the percentage of the variable in y directly attributable to the variation in x. Therefore, if you have a correlation coefficient of 0.8, you can say that approximately 64 per cent (0.82) of the variation in y is explained by variations in x. This figure is known as the **explained variation** whilst the balance of 36% is termed the **unexplained variation**. Unless this unexplained variation is small there may be other causes than the variable x which explain the variation in y, e.g. y may be influenced by other variables or the relationship may be non-linear.

In conclusion, then, the coefficient of linear correlation tells you only part of the nature of the relationship between the variables; it shows that such a relationship exists. You have to interpret the coefficient and use it to deduce the form and find the significance of the association between the variables x and y.

Note on the Computation of r

Often the values of x and y are quite large and the arithmetic involved in calculating r becomes tedious. To simplify the arithmetic and hence reduce the likelihood of numerical slips, it is worth noting the following points:

- a) We can take any constant amount off every value of x
- b) We can take any constant amount off every value of y
- c) We can divide or multiply every value of x by a constant amount
- d) We can divide or multiply every value of y by a constant amount

all without altering the value of r. This also means that the value of r is independent of the units in which x and y are measured.

Let's consider the above example as an illustration. We shall take 5 off all the x values and 2 off all the y values to demonstrate that the value of r is unaffected. We call the new x and y values, x' (xdash) and y' respectively:

Table 11.4

x	y	x	y'	(x') ²	(y') ²	x'y'
10	5	5	3	25	9	15
14	3	9	1	81	1	9
7	5	2	3	4	9	6
12	2	7	0	49	0	0
5	7	0	5	0	25	0
Totals		24	18	160	80	36

$$n = 6$$

$$r = \frac{n \sum x' y' - \sum x' \sum y'}{\sqrt{[n \sum (x')^2 - (\sum x')^2][n \sum (y')^2 - (\sum y')^2]}}$$

$$r = \frac{6 \times 36 - 24 \times 18}{\sqrt{(6 \times 160 - 24^2)(6 \times 80 - 18^2)}}$$

$$r = \frac{216 - 432}{\sqrt{(960 - 576)(480 - 324)}}$$

$$= \frac{-216}{\sqrt{384 \times 156}} = 0.88 \text{ to 2 decimal places.}$$

Thus the result is identical and the numbers involved in the calculation are smaller, taken overall

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D. RANK CORRELATION

General

Sometimes, instead of having actual measurements, we only have a record of the order in which items are placed. Examples of such a situation are:

- a) We may arrange a group of people in order of their heights, without actually measuring them. We could call the tallest No.1, the next tallest No. 2, and so on.
- b) The results of an examination may show only the order of passing, without the actual marks; the highest-marked candidate being No. 1, the next highest being No. 2, and so on.

Data which is thus arranged in order of merit or magnitude is said to be RANKED.

Relationship between Ranked Variates

Consider, as an example, the case of eight students who have taken the same two examinations, one in Mathematics and one in French. We have not been told the actual marks obtained in the examination, but we have been given the relative position (i.e. the RANK) of each student in each subject:

Table 11.5

Student	Relative Position	
	French	Mathematics
A	8	6
B	5	5
C	3	4
D	6	7
E	7	8
F	2	1
G	1	3
H	4	2

We see from this table of ranks that student F was top in Mathematics but only second in French. Student G was top of the class in French, student E was bottom of the class (rank 8) in Mathematics, and so on.

A question which naturally arises is, "Is there any relationship between the students' performances in the two subjects?" This question can be put into statistical terms by asking: "Is there any correlation between the students' ranks in Mathematics and their ranks in French?" The answer to the question will fall into one of the following three categories:

- a) **No correlation:** no connection between performance in the Mathematics examination and performance in the French examination.
- b) **Positive correlation:** students who do well in one of the subjects will, generally speaking, do well in the other.
- c) **Negative correlation:** students who do well in one of the subjects will, generally speaking, do poorly in the other.

We will start our analysis by drawing the scatter diagram as in Figure 11.8. It does not matter which subject we call x and which y.

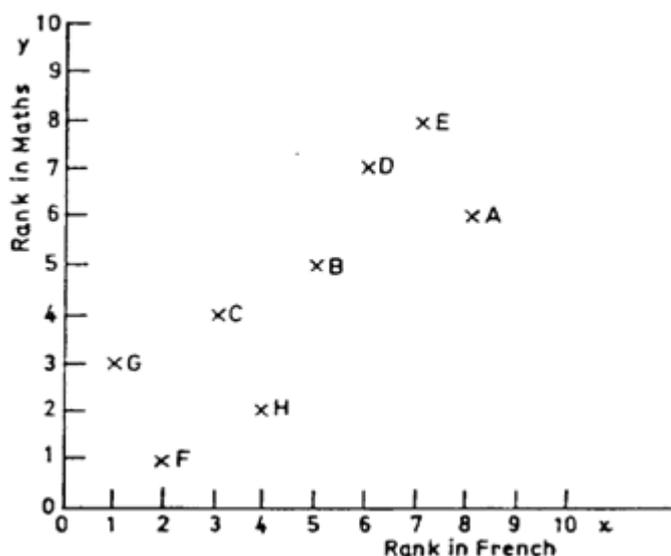


Figure 11.8: Scatter Diagram of Students' Results

The general impression given by the scatter diagram is that there is positive correlation. To find out how strong this correlation is, we calculate the correlation coefficient:

$$r = \frac{n \sum xy - \sum x \sum y}{\sqrt{[n \sum x^2 - (\sum x)^2][n \sum y^2 - (\sum y)^2]}}$$

$$n = 8$$

Table 11.6

Student	Rank in French (X)	Rank in Maths (y)	x^2	y^2	xy
A	8	6	64	36	48
B	5	5	25	25	25
C	3	4	9	16	12
D	6	7	36	49	42
E	7	8	49	64	56
F	2	1	4	1	2
G	1	3	1	9	3
H	4	2	16	4	8
Total	36	36	204	204	196

$$r = \frac{8 \times 196 - (36)^2}{\sqrt{[8 \times 204 - (36)^2][8 \times 204 - (36)^2]}} = \frac{1,568 - 1,296}{1,632 - 1,296} = \frac{272}{336} = 0.81$$

Ranked Correlation Coefficients

With ranked variates, there are simpler methods of calculating a correlation coefficient.

a) Spearman's Rank Correlation Coefficient

This is usually denoted by the letter r_s . Its formula is:

$$r_s = 1 - \frac{6 \sum d^2}{n^3 - n} \quad \text{i.e.} \quad r_s = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}$$

In some books you may find R or the Greek letter ρ (pronounced 'roe') used instead of r_s but you will recognise Spearman's coefficient by its formula.

In this formula, d is the difference between the two ranks for any one item, and n is the number of items involved. In the above example, $n = 8$. You can follow the calculation of r_s in the following table:

Table 11.7

Student	Rank in:		d	d ²
	Maths	French		
A	6	8	-2	4
B	5	5	0	0
C	4	3	1	1
D	7	6	1	1
E	8	7	1	1
F	1	2	-1	1
G	3	1	2	4
H	2	4	-2	4
Total			(Check) 0	16

$$r_s = 1 - \frac{6 \times 16}{8^3 - 8} = 1 - \frac{96}{512 - 8} = 1 - \frac{96}{504} = 1 - \frac{12}{63}$$

$$= 1 - 0.19 = +0.81$$

When there is perfect agreement between the ranks of the two variates, then all the values of d will be 0 and so the rank correlation coefficient will be +1.0. When there is complete disagreement between the ranks, the values of d will be at their maximum and the rank correlation coefficient is -1.0.

b) Kendall's Rank Correlation Coefficient

This is usually denoted by the Greek letter τ (pronounced 'tau'). It does not give exactly the same answer as Spearman's method. Its formula is:

$$\tau = \frac{S}{\frac{1}{2}n(n-1)}$$

where, as before, n is the number of pairs of observations. S is referred to as the score of the ranks. To work out the score, we first arrange the students in order of their French ranks. We then consider for each student in turn whether the differences in French rankings between him and students lower down the list have the same signs as the differences in their Mathematics rankings. If the signs are the same, a pair of students is said to be **concordant**. If the signs are different, the pair is **discordant**. The score, S , is $(n_c - n_d)$ where n_c is the total number of concordant pairs and n_d is the total number of discordant pairs. It is easiest to set out the calculation in a table:

Table 11.8

Student	Rank in:		n_c	n_d	$n_c - n_d$
	French	Mathematics			
G	1	3	5	2	3
F	2	1	6	0	6
C	3	4	4	1	3
H	4	2	4	0	4
B	5	5	3	0	3
D	6	7	1	1	0
E	7	8	0	1	-1
A	8	6	0	0	0
Total					18

Compared with Student G, whose French rank is 1, all other French ranks have a higher numerical value. Student G's Maths rank is 3, however, so there are 5 Maths ranks with a higher numerical value and 2 with a lower numerical value. Thus $n_c = 5$ and $n_d = 2$. Similarly, for Student F, all French ranks below him in the table have higher numerical values and so do all the Maths ranks so $n_c = 6$ and $n_d = 0$. n_c and n_d are found similarly for the other students. Each student should be compared only with those lower down the table, so that each pair of French and Maths rankings is considered once only.

$$\tau = \frac{18}{\frac{1}{2} \times 8 \times 7} = \frac{36}{56} = 0.64$$

This value, being relatively large and positive, again shows a tendency for a high mark in French to be associated with a high mark in Maths, although the agreement is not perfect.

Tied Ranks

Sometimes it is not possible to distinguish between the ranks of two or more items. For example, two students may get the same mark in an examination and so they have the same rank. Or, two or more people in a group may be the same height. In such a case, we give all the equal ones an average rank and then carry on **as if we had given them different ranks**.

You will see what this means by studying the following examples:

- a) First two equal out of eight

1½ 1½ 3 4 5 6 7 8

Average of 1&2

- b) Three equal out of nine, but not at the ends of the list

1 2 3 5 5 5 7 8 9

Average of 1 4,5 & 6

c) Last two equal out of eight

1 2 3 4 5 6 $7\frac{1}{2}$ $7\frac{1}{2}$

Average of 7 & 8

d) Last four equal out of eleven

1 2 3 4 5 6 7 $9\frac{1}{2}$ $9\frac{1}{2}$ $9\frac{1}{2}$ $9\frac{1}{2}$

Average of 8, 9, 10 & 11

Strictly speaking, a rank correlation coefficient should not be used in these cases without making some adjustment for tied ranks. But the formula for the adjustments are a little complex and are outside the scope of this course. The best way for you to deal with tied ranks in practice is to calculate the ordinary (Pearson's) correlation coefficient. If, in an examination, you are specifically asked to calculate a rank correlation coefficient when there are tied ranks, then of course you must do so; but you might reasonably add a note to your answer to say that, because of the existence of tied ranks, the calculated coefficient is only an approximation, although probably a good one.

Final note: Rank correlation coefficients may be used when the actual observations (and not just their rankings) are available. We first work out the rankings for each set of data and then calculate Spearman's or Kendall's coefficient as above. This procedure is appropriate when we require an approximate value for the correlation coefficient. Pearson's method using the actual observations is to be preferred in this case, however, so calculate a rank correlation coefficient only if an examination question specifically instructs you to do so.

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STUDY UNIT 12

Linear Regression

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A. INTRODUCTION

We've seen how the correlation coefficient measures the degree of relationship between two variates. With perfect correlation ($r = +1.0$ or $r = -1.0$), the points of the scatter diagram all lie exactly on a straightline. It is sometimes the case that two variates are perfectly related in some way such that the points would lie exactly on a line, but not a straight line. In such a case r would not be 1.0. This is a most important point to bear in mind when you have calculated a correlation coefficient; the value may be small, but the reason may be that the correlation exists in some form other than a straight line.

The correlation coefficient tells us the extent to which the two variates are linearly related, but it does not tell us how to find the particular straight line which represents the relationship. The problem of determining which straight line best fits the points of a particular scatter diagram comes under the heading of LINEAR REGRESSION analysis.

Remember that a straight-line graph can always be used to represent an equation of the form $y = mx + c$. In such an equation, y and x are the variables while m and c are the constants. Figure 8.1 shows a few examples of straight-line graphs for different values of m and c . Note the following important features of these linear graphs:

- The value of c is always the value of y corresponding to $x = 0$.
- The value of m represents the gradient or slope of the line. It tells us the number of units change in y per unit change in x . Larger values of m mean steeper slopes.

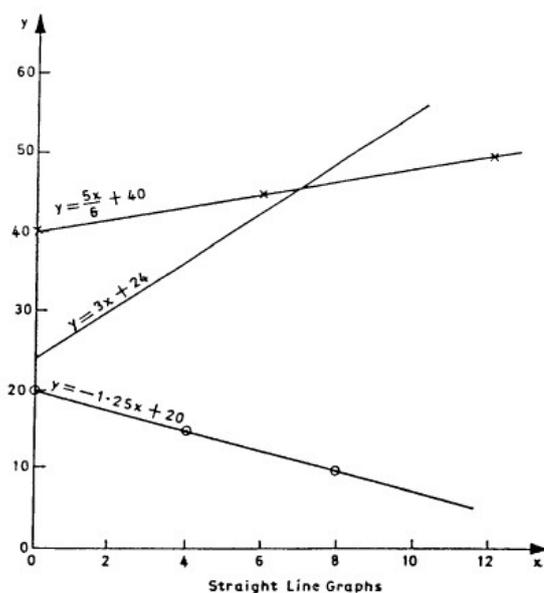


Figure 12.1

- Negative values of the gradient, m , mean that the line slopes downwards to the right; positive values of the gradient, m , mean that the line slopes upwards to the right.

So long as the equation linking the variables y and x is of the form $y = mx + c$, it is always possible to represent it graphically by a straight line. Likewise, if the graph of the relationship between y and x is a straight line, then it is always possible to express that relationship as an equation of the form $y=mx+c$.

Often in regression work the letters a and b are used instead of c and m , i.e. the regression line is written as $y = a + bx$. You should be prepared to meet both forms.

If the graph relating y and x is NOT a straight line, then a more complicated equation would be needed. Conversely, if the equation is NOT of the form $y = mx + c$ (if, for example, it contains terms like x^2 or $\log x$) then its graph would be a curve, not a straight line.

B. REGRESSION LINES

Nature of Regression Lines

When we have a scatter diagram whose points suggest a straight-line relationship (though not an exact one), and a correlation coefficient which supports the suggestion (say, r equal to more than about 0.4 or 0.5), we interpret this by saying that there is a linear relationship between the two variables but there are other factors (including errors of measurement and observation) which operate to give us a scatter of points around the line instead of exactly on it.

In order to determine the relationship between y and x , we need to know what straight line to draw through the collection of points on the scatter diagram. It will not go through all the points, but will lie somewhere in the midst of the collection of points and it will slope in the direction suggested by the points. Such a line is called a REGRESSION LINE.

In Figure 12.2 x is the monthly output of a factory and y is the total monthly costs of the factory; the scatter diagram is based on last year's records. The line which we draw through the points is obviously the one which we think best fits the situation, and statisticians often refer to regression lines as lines of best fit. Our problem is how to draw the best line.

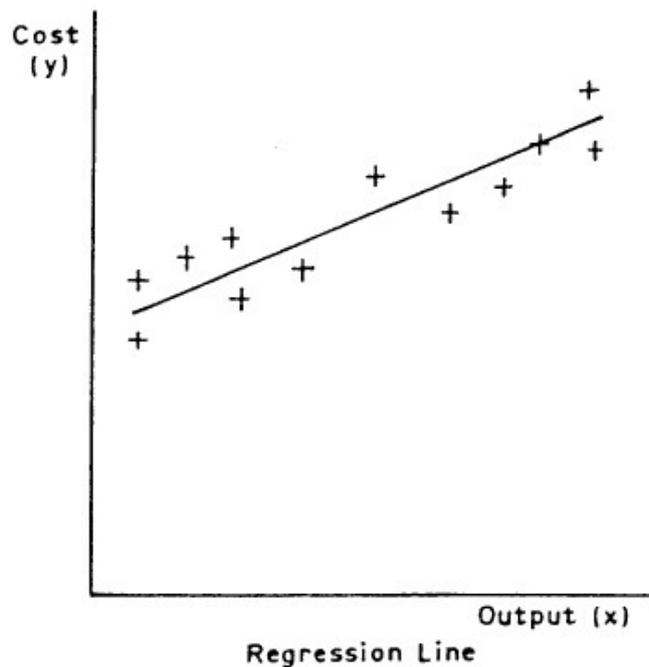


Figure 12.2

There are two methods available - a graphical method and a mathematical method.

Graphical Method

It can be proved mathematically (but you don't need to know how!) that the regression line must pass through the point representing the arithmetic means of the two variables. The graphical method makes use of this fact, and the procedure is as follows:

- a) Calculate the means and of the two variables.
- b) Plot the point corresponding to this pair of values on the scatter diagram.
- c) Using a ruler, draw a straight line through the point you have just plotted and lying, as evenly as you can judge, among the other points on the diagram.

In Figure 12.3 the above procedure was followed using the data from the section on the correlation coefficient in the previous study unit. If someone else (you, for example) were to do it, you might well get a line of a slightly different slope, but it would still go through the point of the means (marked +).

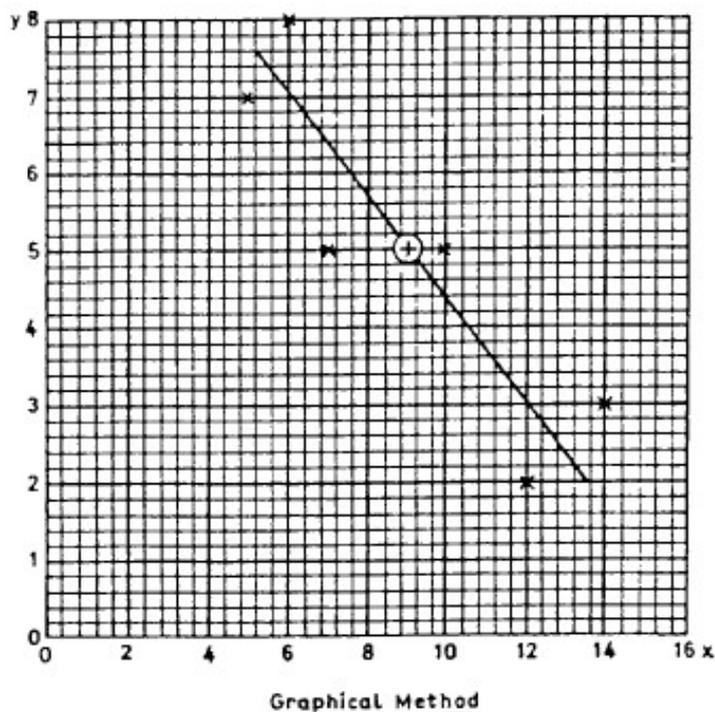


Figure 12.3

Quite obviously, this method is not exact (no graphical methods are) but it is often sufficient for practical purposes. The stronger the correlation, the more reliable this method is, and with perfect correlation there will be little or no error involved.

Mathematical Method

A more exact method of determining the regression line is to find mathematically the values of the constants m and c in the question $y = mx + c$, and this can be done very easily. This method is called the **least squares** method, as the line we obtain is that which **minimises the sum of the squares of the vertical deviations of the points from the line**. The equation of the least squares line is:

$$y = mx + c \text{ although this is sometimes written as } y = a + bx$$

$$\text{where } m = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

$$c = \bar{y} - m\bar{x} \text{ or } \frac{\sum y - m \sum x}{n}$$

n = number of pairs of readings.

We will now apply these formulae to the example we used when talking about the correlation coefficient. If you look back at the last study unit you will see that we had the following figures:

$$\sum x = 54; \quad \sum y = 30; \quad \sum x^2 = 550 \quad \sum xy = 234; \quad n=6$$

$$\therefore \bar{x} = 9$$

$$\text{and } \bar{y} = 5$$

Applying the formulae, we get:

$$m = \frac{6 \times 234 - 54 \times 30}{6 \times 550 - (54)^2} = \frac{-216}{384} = -0.5625$$

$$c = 5 - (-0.5625)9 = 5 + 5.0625 = 10.0625$$

m and c are termed the **regression coefficients** (and m also represents the gradient, as previously stated). The equation for the regression line in this case is therefore:

$$y = 10.0625 - 0.5625x$$

To draw this line on the scatter diagram, choose two values of x, one towards the left of the diagram and one towards the right. Calculate y for each of these values of x, plot the two points and join them up with a straight line. If you have done the calculations correctly, the line will pass through the (\bar{x}, \bar{y}) point.

For drawing the regression line, we will choose values of x which are convenient, e.g. $x = 0$ and $x = 16$. The corresponding values of y are:

$$\text{For } x = 0, \quad y = 10.0625 - 0 = 10.0625$$

$$\begin{aligned} \text{For } x = 16, \quad y &= 10.0625 - 16(0.5625) = 10.0625 - 9.0 \\ &= 1.0625 \end{aligned}$$

The two points marked . are shown in the scatter diagram in Figure 8.4, together with the individual points (x), the regression line (drawn as an unbroken line) and the mean point (+).

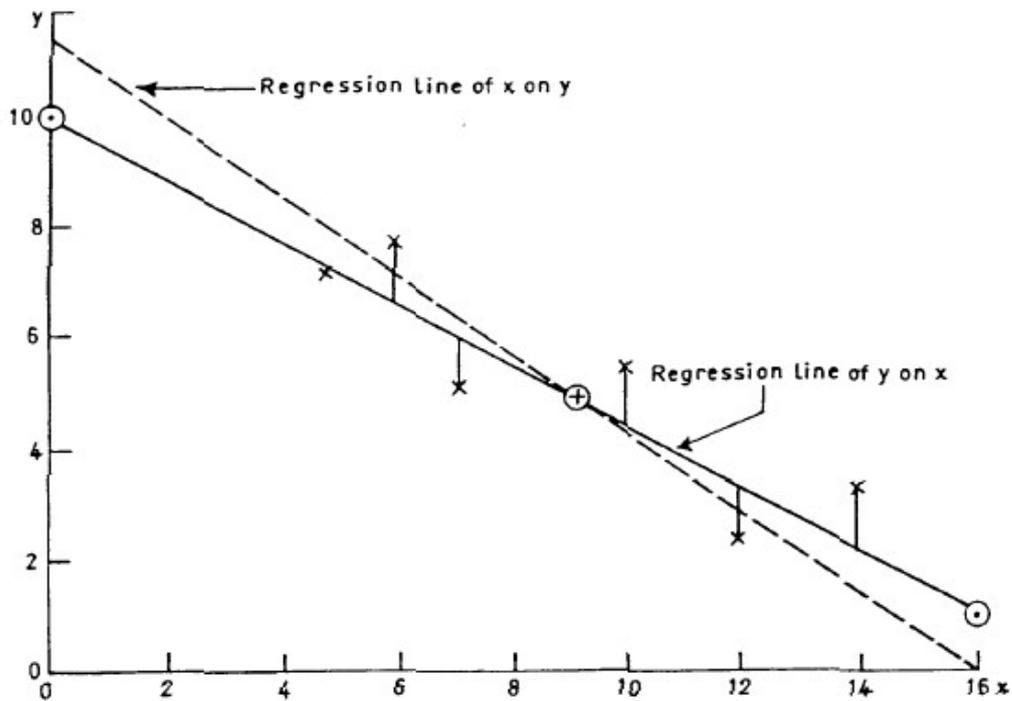


Figure 12.4

The regression line which we have drawn, and the equation which we have determined, represent the **regression of y upon x**. We could, by interchanging x and y, have obtained the regression of x on y. This would produce a different line and a different equation. This latter line is shown in Figure 8.4 by a broken line. The question naturally arises, "Which regression line should be used?". The statistician arrives at the answer by some fairly complicated reasoning but, for our purposes, the answer may be summed up as follows:

- a) Always use the regression of y on x. That is, use the method described in detail above, putting y on the **vertical** axis and x on the **horizontal** axis.
- b) If you intend to use the regression line to predict one thing from another, then the thing you want to predict is treated as y; the other thing is x. For example, if you wish to use the regression line (or its equation) to predict costs from specified outputs, then the outputs will be the x and the costs will be the y.
- c) If the regression is not to be used for prediction, then the x should be the variate whose value is known more reliably.

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C. USE OF REGRESSION

The main use of a regression line is to calculate values of the dependent variable not observed in the data set. Take as our example that of employees' heights with a regression equation of $y = 2.87(x) - 345.33$ where x is height. Of the 12 people measured and weighed there was nobody of height 181 cm; therefore, if we wanted to know the weight of somebody of this height, it would be impossible to read it from the data available. However, by assuming that a linear relationship exists between weight and height it is possible, by using the regression equation, to calculate an estimate of the weight:

$$\begin{aligned}x = 181 \quad y &= 2.87(181) - 345.33 \\ &= 174.14 \text{ Ib}\end{aligned}$$

Therefore the estimated weight of somebody of height 181 cm is 174.14 Ib.

Since the value of x (181 cm) lies **within the observed range** of x from the 12 people, we say that we have estimated the value of y by **interpolation**.

However, if we wish to use a regression equation to forecast a result from values which are **outside the range of observations** from which the line is calculated, we have to consider carefully the validity of the estimate obtained. This use of the regression line is called **extrapolation** and we have to assume that the same linear relationship will exist for observations beyond those from which it has been formulated. For example, say we want to estimate the weight of somebody whose height is 194 cm, this value is outside the range of the 12 people measured but y can still be calculated as:

$$\begin{aligned}x = 194 \quad \therefore \quad y &= 2.87(194) - 345.33 \\ y &= 211.45 \text{ lb}\end{aligned}$$

This result seems reasonable, but common sense suggests that values of x much smaller than 160 cm or much larger than 186 cm would be rather improbable.

Sometimes this assumption of the same linear relationship is incorrect, as the factors that influenced the two variables may not remain constant outside the range from which the regression equation is formed, or some extra factor may be introduced.

Consider the relationship between time and the average working wage; if a regression line calculated from data that is collected during years where inflation is very low is used to

estimate the wage for years of high inflation, the predicted figure will be much lower than the actual figure, i.e. the change in inflation will change the relationship between the variables. This emphasises that extrapolation gives reliable results only for values **close to the ends of the observed range**.

D. CONNECTION BETWEEN CORRELATION AND REGRESSION

The degree of correlation between two variables is a good guide to the likely accuracy of the estimates made from the regression equation. If the correlation is high then the estimates are likely to be reasonably accurate, and if the correlation is low then the estimates will be poor as the unexplained variation is then high.

You must remember that both the regression equations and the correlation coefficient are calculated from the same data, so both of them must be used with caution when estimates are predicted for values outside the range of the observations, i.e. when values are predicted by extrapolation or the correlation coefficient is assumed to remain constant under these conditions. Also remember that the values calculated for both correlation and regression are influenced by the number of pairs of observations used. So results obtained from a large sample are more reliable than those from a small sample.

Questions on correlation and regression are frequently set in examinations and they are also in practical use in many business areas. Therefore a thorough knowledge of both topics is important.

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STUDY UNIT 13

Time Series Analysis I

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A. INTRODUCTION

Businesses and governments use statistical analysis of information collected at regular intervals over extensive periods of time to plan future policies. For example, sales values or unemployment levels recorded at yearly, quarterly or monthly intervals are examined in an attempt to predict their future behaviour. Such sets of values observed at regular intervals over a period of time are called time series.

The analysis of this data is a complex problem as many variable factors may influence the changes. The first step is to plot the observations on a scattergram, which differs from those we have considered previously, as the points are evenly spaced on the time axis in the order in which they are observed, and the time variable is always the independent variable. This scattergram gives us a good visual guide to the actual changes but is very little help in showing the component factors causing these changes or in predicting future movements of the dependent variable.

Statisticians have constructed a number of mathematical models to describe the behaviour of time series, and several of these will be discussed in this study unit and the next.

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B. STRUCTURE OF A TIME SERIES

These models assume that the changes are caused by the variation of four main factors dealt with below they differ in the relationship between these factors. It will be easier to understand the theory in detail if we relate it to a simple time series so that we can see the calculations necessary at each stage.

Consider a factory employing a number of people in producing a particular commodity, say thermometers. Naturally, at such a factory during the course of a year some employees will be absent for various reasons. The following table shows the number of days lost through sickness over the last five years. Each year has been broken down into four quarters of three months. We have assumed that the number of employees at the factory remained constant over the five years.

Year	Quarter	Days Lost
19.2	1	30
	2	20
	3	15
	4	35
19.3	1	40
	2	25
	3	18
	4	45
19.4	1	45
	2	30
	3	22
	4	55
19.5	1	50
	2	32
	3	28
	4	60
19.6	1	60
	2	35
	3	30
	4	70

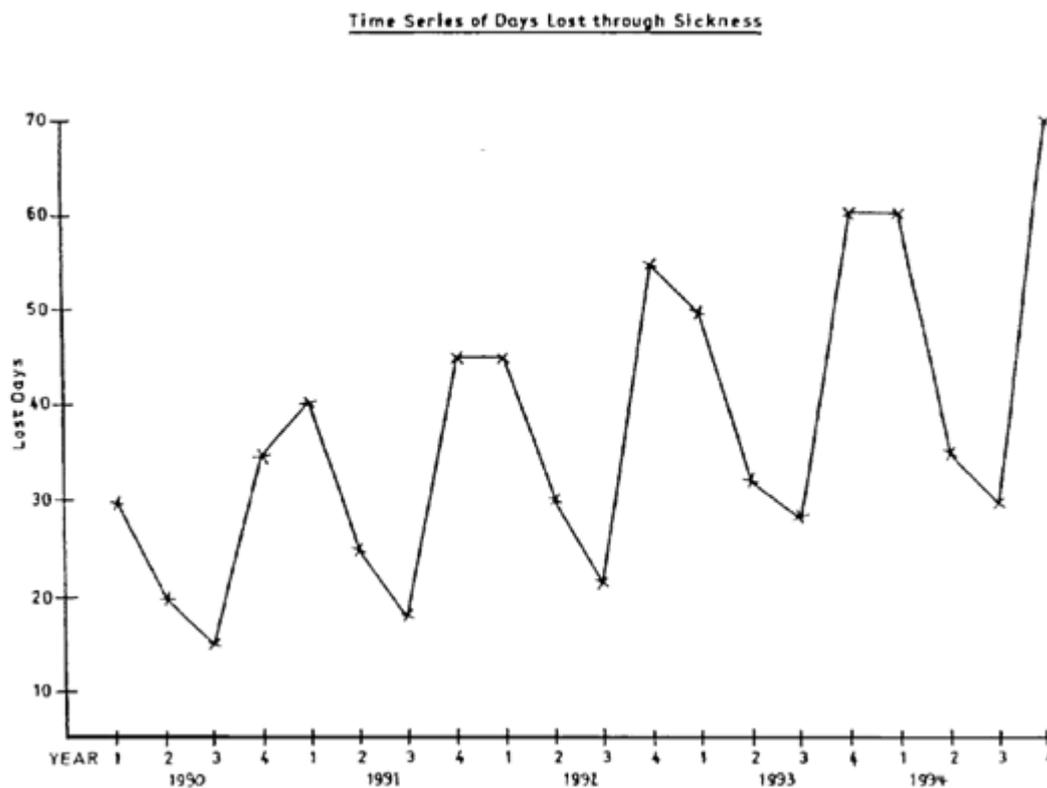
Table 13.1

We will begin by plotting the scattergram for the data, as shown in Figure 13.2.

The scattergram of a time series is often called a **historigram**. (Do not confuse this with a histogram, which is a type of bar chart.) Note the following characteristics of a historigram:

- a) It is usual to join the points by straight lines. The only function of these lines is to help your eyes to see the pattern formed by the points.
- b) Intermediate values of the variables cannot be read from the histogram.
- c) A histogram is simpler than other scattergrams since no time value can have more than one corresponding value of the dependent variable.
- d) Every histogram will look similar to this, but a careful study of the change of pattern over time will suggest which model should be used for analysis.

Figure 13.2



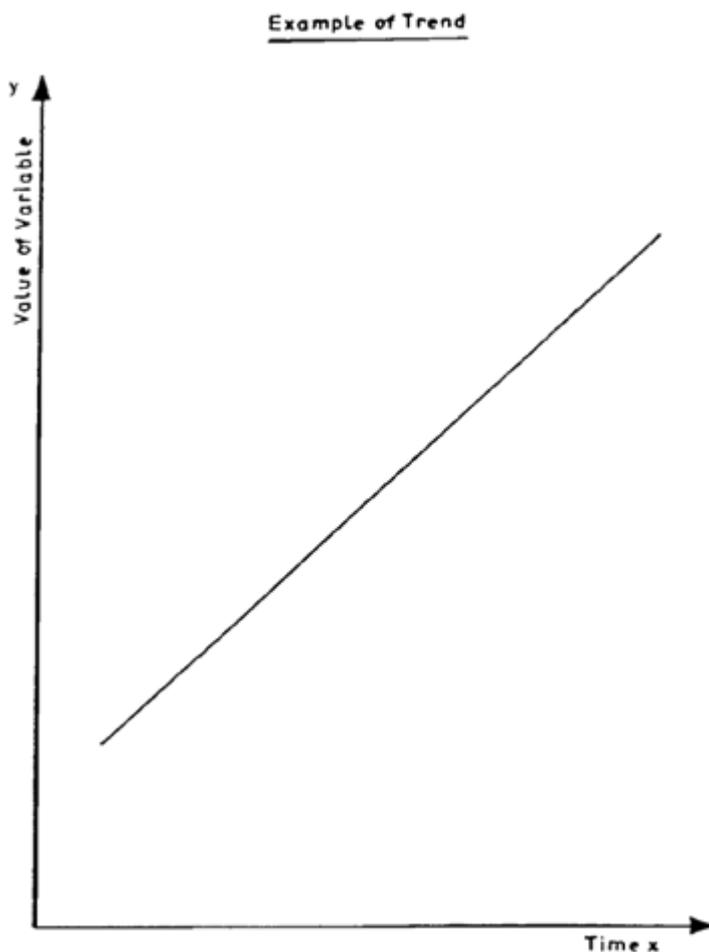
There are four factors that influence the changes in a time series - trend, seasonal variations, cyclical fluctuations, irregular or random fluctuations. Now we will consider each in turn.

Trend

This is the change in general level over the whole time period and is often referred to as the **secular trend**. You can see in Figure 9.1 that the trend is definitely upwards, in spite of the obvious fluctuations from one quarter to the next.

A trend can thus be defined as a **clear tendency for the time series data to travel in a particular direction** in spite of other large and small fluctuations. An example of a linear trend is shown in Figure 13.3. There are numerous instances of a trend, for example the amount of money collected from Rwandan taxpayers is always increasing; therefore any time series describing income from tax would show an upward trend.

Figure 13.3



Seasonal Variations

These are variations which are repeated over relatively short periods of time. Those most frequently observed are associated with the seasons of the year, e.g. ice-cream sales tend to rise during the summer months and fall during the winter months. You can see in our example of employees' sickness that more people are sick during the winter than in the summer.

If you can establish the variation throughout the year then this **seasonal variation** is likely to be similar from one year to the next, so that it would be possible to allow for it when estimating values of the variable in other parts of the time series. The usefulness of being able to calculate seasonal variation is obvious as, for example, it allows ice-cream manufacturers to alter their production schedules to meet these seasonal changes. Figure 13.4 shows a typical seasonal variation that could apply to the examples above.

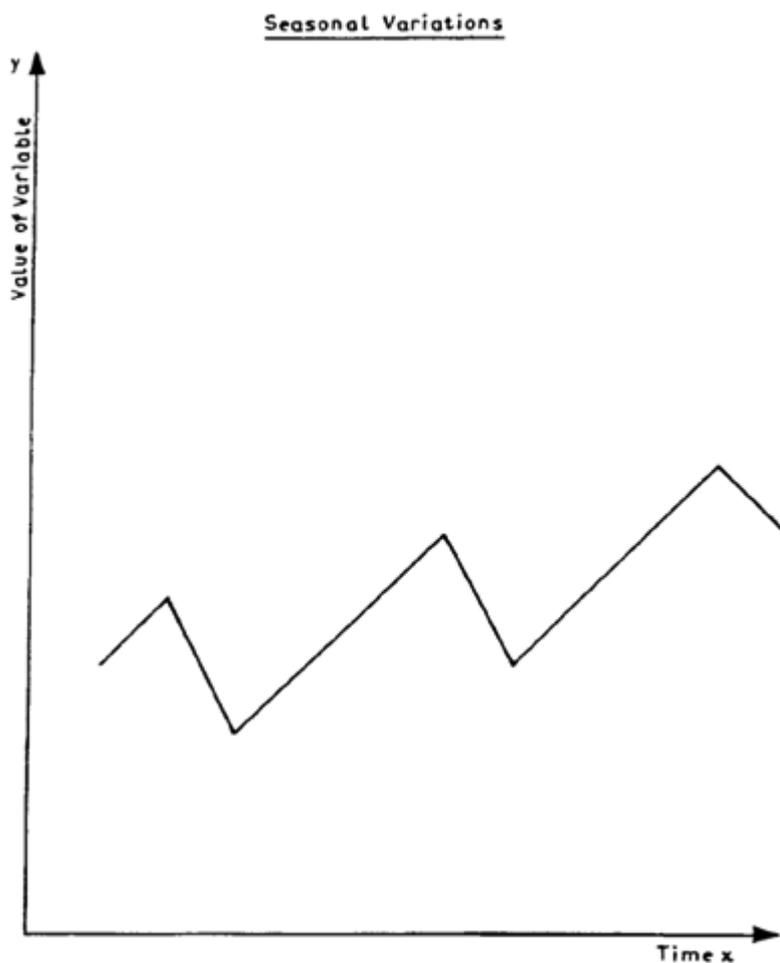


Figure 13.4

Cyclical Fluctuations

These are long-term but fairly regular variations. They are difficult to observe unless you have access to data over an extensive period of time during which external conditions have remained relatively constant. For example, it is well known in the textile trade that there is a cycle of about three years, during which time demand varies from high to low. This is similar to the phenomena known as the trade cycle which many economists say exists in the trading pattern of most countries but for which there is no generally accepted explanation.

Figure 13.5 shows how such a cyclical fluctuation would relate to an upward trend. In our example on sickness, a cyclical fluctuation could be caused by, say, a two-year cycle for people suffering from influenza.

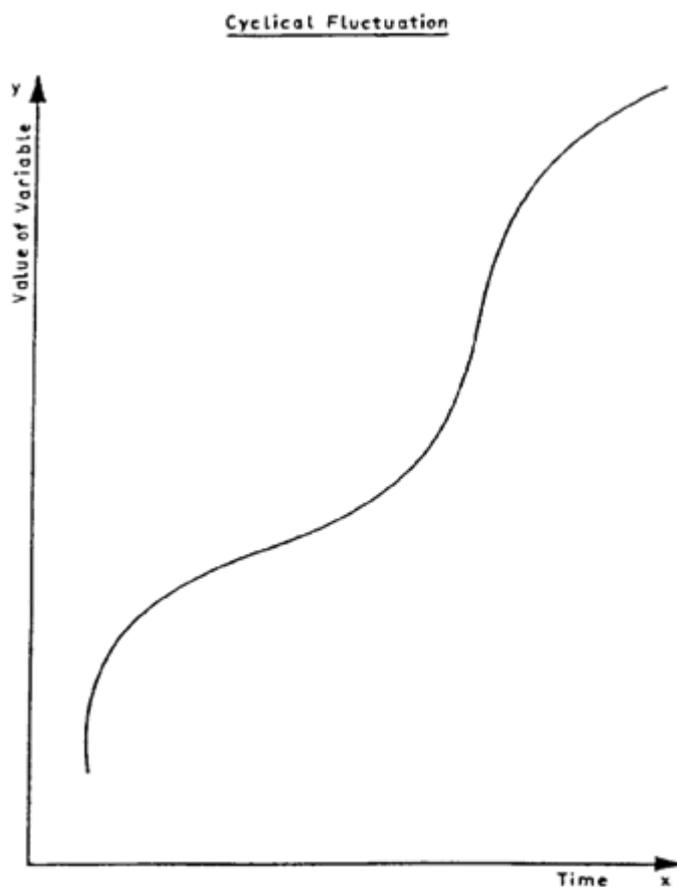


Figure 13.5

As this type is difficult to determine, it is often considered with the final (fourth) element, and the two together are called the residual variation.

Irregular or Random Fluctuations

Careful examination of Figure 9.1 shows that there are other relatively small irregularities which we have not accounted for and which do not seem to have any easily seen pattern. We call these irregular or random fluctuations and they may be due to errors of observation or to some one-off external influence which is difficult to isolate or predict. In our example there may have been a measles epidemic in 19.5, but it would be extremely difficult to predict when and if such an epidemic would occur again.

Summary

To sum up, a time series (Y) can be considered as a combination of the following four factors:

Trend (T)

Seasonal variation (S)

Cyclical fluctuation (C)

Irregular fluctuations (I)

It is possible for the relationship between these factors and the time series to be expressed in a number of ways through the use of different mathematical models. We are now going to look in detail at the **additive model** and in the next study unit we will cover briefly the multiplicative and logarithmic models. The additive model can be expressed by the equation:

$$\text{Time Series} = \text{Trend} + \text{Seasonal Variation} + \text{Cyclical} \\ \text{Fluctuations} + \text{Random Fluctuations}$$

$$\text{i.e. } Y = T + S + C + I$$

Usually the cyclical and random fluctuations are put together and called the 'residual' (R),

$$\text{i.e. } Y=T+S+R$$

C. CALCULATION OF COMPONENT FACTORS FOR THE ADDITIVE MODEL

Trend

The **most important factor** of a time series is the trend, and before deciding on the method to be used in finding it, we must decide whether the conditions that have influenced the series have remained stable over time. For example, if you have to consider the production of some commodity and want to establish the trend, you should first decide if there has been any significant change in conditions affecting the level of production, such as a sudden and considerable growth in the national economy. If there has, you must consider breaking the time series into sections over which the conditions have remained stable.

Having decided the time period you will analyse, you can use any one of the following methods to find the trend. The basic idea behind most of these methods is to average out the three other factors of variation so that you are left with the long-term trend.

a) Graphical Method

Once you have plotted the histogram of the time series, it is possible to draw in by eye a line through the points to represent the trend. The result is likely to vary considerably from person to person, unless the plotted points lie very near to a straight line, so it is not a satisfactory method.

b) Semi-Averages Method

This is a simple method which involves very little arithmetic. The time period is divided into equal parts, and the arithmetic means of the values of the dependent variable in each half are calculated. These means are then plotted at the quarter and three-quarters position of the time series. The line adjoining these two points represents the trend of the series. Note that this line will pass through the overall mean of the values of the dependent variable. In our example which consists of five years of data, the midpoint of the whole series is mid-way between quarter 2 and quarter 3 of 19.4.

For the mean of the first half:

Year and Quarter		No of days
19.2	1	30
	2	20
	3	15
	4	35
19.3	1	40
	2	25
	3	18
	4	45
19.4	1	45
	2	<u>30</u>
Total		<u>303</u>

Mean = 30.3

Table 13.6

For the mean of the second half:

Year and Quarter		No. of days
19.4	3	22
	4	55
19.5	1	50
	2	32
	3	28
	4	60
19.6	1	60
	2	35
	3	30
	4	<u>70</u>
Total		<u>442</u>

Mean = 44.2

Table 13.7

These values are plotted on the histogram in Figure 13.8. You will notice that 30.3 days, as it is the mean for the first half, is plotted halfway between quarters 1 and 2 of 19.3, and likewise 44.2 days is plotted halfway between quarters 3 and 4 of 19.5. The trend line is then

drawn between these two points and it can be extrapolated beyond these points as shown by the dotted line.

If there is an odd number of observations in the time series, the middle observation is ignored and the means of the observations on each side of it are calculated.

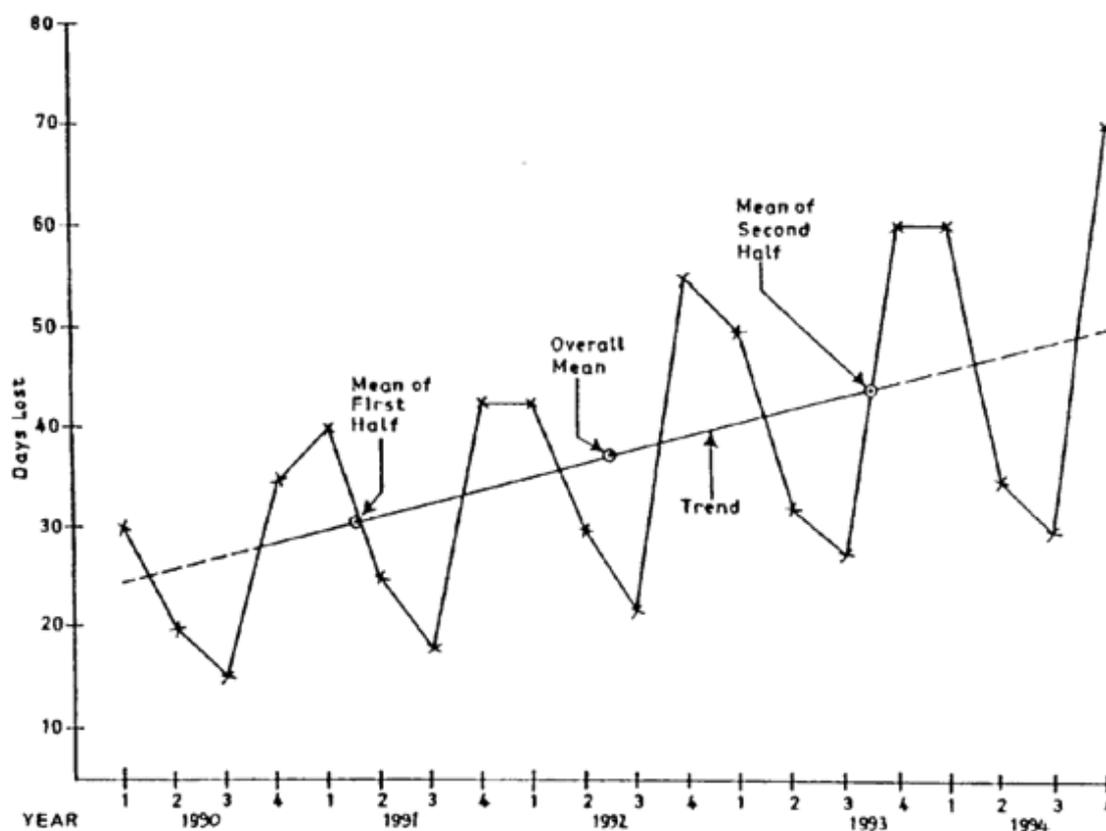


Figure 13.8

c) Least Squares Method

The trend line is calculated using the formula in Study Unit 8 Section B. In fact the trend line is the regression line of y on x where y is the dependent variable and x is the time variable. Since in a time series the observations are always recorded at equally-spaced time intervals, we can represent x by the first n positive integers, where n is the number of observations. We never calculate the other regression line in time series analysis as it has no significance. Thus the equation of the trend is:

$$y = a + bx \quad (1)$$

$$b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} \quad (2)$$

$$a = \frac{\sum y - b \sum x}{n} \quad (3)$$

Using the data given in our earlier example we set up a table of calculations as follows:

Year	Quarter	x	Days Lost y	x ²	xy
19.2	1	1	30	1	30
	2	2	20	4	40
	3	3	15	9	45
	4	4	35	16	140
19.3	1	5	40	25	200
	2	6	25	36	150
	3	7	18	49	126
	4	8	45	64	360
19.4	1	9	45	81	405
	2	10	30	100	300
	3	11	22	121	242
	4	12	55	144	660
19.5	1	13	50	169	650
	2	14	32	196	448
	3	15	28	225	420
	4	16	60	256	960
19.6	1	17	60	289	1,020
	2	18	35	324	630
	3	19	30	361	570
	4	<u>20</u>	<u>70</u>	<u>400</u>	<u>1,400</u>
Σ = <u>210</u>			<u>745</u>	<u>2,870</u>	<u>8,796</u>

$$n = 20$$

$$\therefore b = \frac{20(8,796) - 210(745)}{20(2,870) - (210)^2} = \frac{175,920 - 156,450}{57,400 - 44,100}$$

$$= \frac{19,470}{13,300} = 1.46$$

$$\therefore a = \frac{745 - 1.46(210)}{20} = \frac{438.4}{20} = 21.92$$

So the equation of the trend line is:

$$y = 21.92 + 1.46x$$

where y is the number of days lost owing to sickness and x is the number given to the quarter required. We can now draw the line represented by this equation on the time series histogram as shown in Figure 9.6. This method uses all the available information, but it suffers from the same limitations as other regression lines if it is used for prediction by extrapolation.

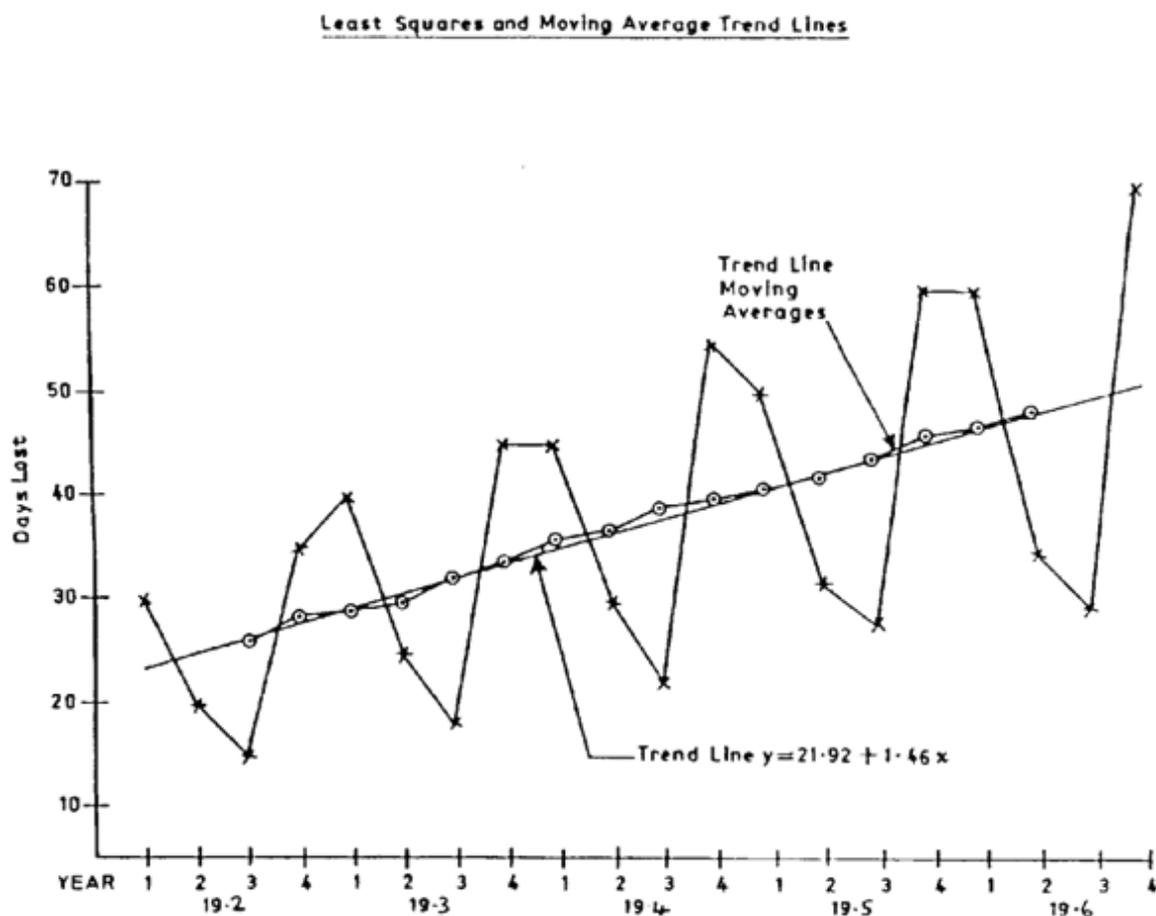


Figure 13.9

d) Moving Averages Method

So far, the methods we have discussed for finding trends have resulted in a straight line, but the actual trend may be a curve or a series of straight segments. The method of moving averages gives a way of calculating and plotting on the histogram a trend point corresponding to each observed point. These points are calculated by averaging a number of consecutive values of the dependent variable so that variations in individual observations are reduced. The number of consecutive values selected will depend on the length of the short-term or seasonal variation shown on the histogram.

The method of calculating a set of moving averages is illustrated by the following simple example. Consider the seven numbers 6, 4, 5, 1, 9, 5, 6 and take the number of time periods covered by the fluctuations to be four as in quarterly figures, then a moving average of order four is needed.

Step 1: Find the average of the first to fourth numbers.

$$\text{Average} = \frac{6 + 4 + 5 + 1}{4} = 4$$

Step 2: Find the average of the second to fifth numbers.

$$\text{Average} = \frac{4 + 5 + 1 + 9}{4} = 4.75$$

Step 3: Find the average of the third to sixth numbers.

$$\text{Average} = \frac{5 + 1 + 9 + 5}{4} = 5$$

Step 4: Find the average of the fourth to seventh numbers.

$$\text{Average} = \frac{1 + 9 + 5 + 6}{4} = 5.25$$

Hence the moving averages of order 4 are 4, 4.75, 5, 5.25. For monthly data a moving average of order 12 would be needed; for daily data the order would be 7, and so on.

Using the data of the earlier example, we calculate the trend values and plot them on Figure 13.9 so that we can compare the two trend lines. The table of calculations follows:

Year	Quarter	Days Lost	4-Quarter Total	Moving Average	Trend
(1)	(2)	(3)	(4)	(5)	(6)
19.2	1	30			
	2	20			
	3	15	100	25	26.3
	4	35	110	27.5	28.1
19.3	1	40	115	28.75	29.1
	2	25	118	29.5	30.8
	3	18	128	32.0	32.6
	4	45	133	33.25	33.9
19.4	1	45	138	34.5	35.0
	2	30	142	35.5	36.8
	3	22	152	38.0	38.6
	4	55	157	39.25	39.5
19.5	1	50	159	39.75	40.5
	2	32	165	41.25	41.9
	3	28	170	42.5	43.8
	4	60	180	45.0	45.4
			183	45.75	

Table 13.10

The trend is given correct to one decimal place as this is the greatest accuracy justified by the accuracy of the data. Notice how the table of calculations is set out, with the numbers in columns (4) and (5) placed **midway between** two quarterly readings. This is because we were averaging over an even number of values, so the moving average would have to be plotted in this position on the histogram and would not correspond to any particular quarter.

Thus it is necessary to add column (6) which gives the mean of successive pairs of moving averages and these numbers are the trend values plotted. (The values in column (6) are often called the **centred moving averages**.)

If we were calculating a moving average with an odd number of values it would not be necessary to carry out this final stage as the moving averages would be centred on an actual observation and so would be the trend values, e.g. daily observation over a number of weeks or data with a short-term cycle of an odd number of years.

The main advantage of this method is that the trend values take into account the **immediate** changes in external factors which the trend lines, using the previous two methods, are unable to do. However, this method has three disadvantages:

- (i) The trend line cannot be found for the whole of the time series. As you can see from our example, there are no trend values for quarters at the beginning and end of the series.
- (ii) Problems can be encountered in deciding the order number, i.e. the period of fluctuation. Unless the seasonal or cyclical movement is definite and clear cut, the moving method of deriving the trend may yield a rather unsatisfactory line.
- (iii) Since the trend is calculated as a simple arithmetic mean it can be unduly influenced by a few extreme values.

Seasonal Variation

As we are assuming in this study unit that the additive model is satisfactory, once we have found the trend by one of the methods described in the previous section we can find the value of the remaining factors for each value of the dependent variable from the equation for the additive model by subtraction:

$$\text{i.e. } Y = T + S + C + I$$

$$\text{so } Y - T = S + C + I = S + R$$

$$(C + I = R \text{ since we cannot usually separate } C \text{ and } I)$$

Column (5) of the following table shows the value of this difference for all the quarters from 19.2 quarter 3 to 19.6 quarter 2.

Year (1)	Quarter (2)	Days Lost (Y) (3)	Trend (T) (4)	Y -T (5)
19.2	3	15	26.3	-11.3
	4	35	28.1	6.9
19.3	1	40	29.1	10.9
	2	25	30.8	-5.8
	3	18	32.6	-14.6
	4	45	33.9	11.1
19.4	1	45	35.0	10.0
	2	30	36.8	-6.8
	3	22	38.6	-16.6
	4	55	39.5	15.5
19.5	1	50	40.5	9.5
	2	32	41.9	-9.9
	3	28	43.8	-15.8
	4	60	45.4	14.6
19.6	1	60	46.0	14.0
	2	35	47.5	-12.5

Table 13.11

One of the assumptions we make for the additive model is that the seasonal variations are the same for corresponding quarters in each year. You can see that this is not the case in column (5) except that for each year the first and fourth quarters give a positive result and the second and third a negative one. The variation must be caused by the residual (R), and this factor can be eliminated by calculating the adjusted average for each quarter as shown in the next table:

Table 13.12

Year	1st Qtr	2nd Qtr	3rd Qtr	4th Qtr	
19.2	-	-	-11.3	6.9	
19.3	10.9	-5.8	-14.6	11.1	
19.4	10.0	-6.8	-16.6	15.5	
19.5	9.5	-9.9	-15.8	14.6	
19.6	<u>14.0</u>	<u>-12.5</u>	-	-	
Total	<u>44.4</u>	<u>-35.0</u>	<u>-58.3</u>	<u>48.1</u>	
Average	11.1	-8.8	-14.6	12.0	(-0.3)
Adjusted Average	11.175	-8.725	-14.525	12.075	

The average fluctuations should add up to zero, but as you can see in the example above, because of rounding errors they do not; therefore a minor adjustment is carried out in the last row. This is done by subtracting a quarter of the total outstanding from each average (in this case 0.25 of $-0.3 = -0.075$).

Therefore the values 11.2, -8.7, -14.5 and 12.1 (all correct to 1 dp) are the seasonal fluctuations of the four quarters for the time series of days lost through sickness at a factory.

Deseasonalised Data and Residual

The remaining results that are needed for this analysis are the deseasonalised values ($Y - S$) and the residuals ($Y - S - T$). These are shown in columns (4) and (6) of the following table:

Year and Qtr		Days Lost	Seasonal Adjustment	Deseasonalised Data	Trend	Residual
(1)	(2)	(3)	(4)	(5)	(6)	$R = Y - S - T$
19.2	3	15	-14.5	29.5	26.3	3.2
	4	35	12.1	22.9	28.1	-5.2
19.3	1	40	11.2	28.8	29.1	-0.3
	2	25	-8.7	33.7	30.7	3.0
	3	18	-14.5	32.5	32.6	-0.1
	4	45	12.1	32.9	33.9	-1.0
19.4	1	45	11.2	33.8	35.0	-1.2
	2	30	-8.7	38.7	36.7	2.0
	3	22	-14.5	36.5	38.6	-2.1
	4	55	12.1	42.9	39.5	3.4
19.5	1	50	11.2	38.8	40.5	-1.7
	2	32	-8.7	40.7	41.9	-1.2
	3	28	-14.5	42.5	43.7	-1.2
	4	60	12.1	47.9	45.4	2.5
19.6	1	60	11.2	48.8	46.0	2.8
	2	35	-8.7	43.7	47.5	-3.8

Table 13.13

As you can see, there is no pattern to the residuals but they are fairly small, i.e. they can be considered as random errors of observation and rounding, though they may contain a systematic cyclic element.

In the next study unit we will look at other time series models and methods used to forecast future values in the series.

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STUDY UNIT 14

Time Series Analysis II

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A. FORECASTING

Assumptions

The reason for isolating the trend within a time series is to be able to make a prediction of its future values and thus estimate the movement of the time series. Before looking at the various methods available to carry out this process, we must state two assumptions that must be made when forecasting:

a) That Conditions Remain Stable

Those conditions and factors which were apparent during the period over which the trend was calculated must be assumed to be unchanged over the period for which the forecast is made. If they do change, then the trend is likely to change with them, thus making any predictions inaccurate, e.g. forecasts of savings trends based on given interest rates will not be correct if there is a sudden change either up or down in these rates.

b) That Extra Factors Will Not Arise

It is sometimes the case that, when trends are predicted beyond the limits of the data from which they are calculated, extra factors will arise which influence the trend. For example, there is a limit to the number of washing machines that can be sold within a country. This capacity is a factor that must be considered when making projections of the future sales of washing machines. Therefore, in forecasting from a time series it must be assumed that such extra factors will not arise.

These assumptions are similar to those mentioned when we looked at the extrapolation of a regression line.

Methods of Forecasting

There are two main methods of forecasting, although both are primarily concerned with short-term forecasts because the assumptions mentioned previously will break down gradually for periods of longer than about a year.

a) Moving Averages Method

This method involves extending the moving average trend line drawn on the histogram of the time series. The trend line is extended by assuming that the gradient remains the same as that calculated from the data. The further forward you extend it, the more **unreliable** becomes the forecast.

When you have read the required trend value from the graph, the appropriate seasonal fluctuation is added to this and allowance is made for the residual variation. For example, consider the premium bond sales in the United Kingdom shown in Figure 14.1. On this figure the moving average trend line stops at the second quarter of 19.6. If this line is extrapolated with the same gradient to the first quarter of 19.7 then:

$$19.7\text{1st Qtr} \quad \text{Trend} = 750$$

This is multiplied by the seasonal variation as it is a multiplicative model, i.e. $750 \times 1.49 = 1,118$, and the residual variation which varied by as much as + 18% is added to this. Therefore the final short-term estimate for the sales of premium bonds for the first quarter of 19.7 is RWF1,118,000 \pm RWF201,000.

Although fairly easy to calculate, this forecast, like all others, must be treated with caution, because it is based on the value of the trend calculated for the second quarter of 19.6, so if this happens to be an especially high or low value then it would influence the trend, and thus the forecast, considerably.

b) Least Squares Method

If the line of best fit, $y = a + bx$, is used as the trend line and drawn on a histogram, it can be extended to give an estimate of the trend. Preferably the required value of x can be substituted in the equation to give the trend value. The seasonal fluctuation and residual variations must be added as in (a).

Using the results of the example from the previous study unit involving days lost through sickness at a factory, the trend line was:

$$y = 21.92 + 1.46x$$

where x took all the integer values between 1 and 20. ,

Now suppose we want to estimate the number of days lost in the first quarter of 19.7, i.e. when $x = 21$. The value of the trend would be:

$$\begin{aligned}y &= 21.92 + 1.46(21) \\ &= 52.58 \\ &= 53 \text{ days}\end{aligned}$$

(This result could also be read from the graph in Figure 13.9)

To this must be added, as it is an additive model, the seasonal fluctuation for a first quarter, which was about 11 days, making a total of 64 days. The residual variation for this series was a maximum of + 5 days. Therefore the forecast for days lost through sickness for the first quarter of 19.7 is between 59 and 69 days.

This forecast again is not entirely reliable, as the trend is depicted by one straight line of a fixed gradient. It is a useful method for short-term forecasting, although like the previous method it becomes more **unreliable** the further the forecast is extended into the future.

There are no hard and fast rules to adopt when it comes to choosing a forecast method. Do not think that the more complicated the method the better the forecast. It is often the case that the simpler, more easily understood methods produce better forecasts, especially when you consider the amount of effort expended in making these forecasts. Remember that, whatever the method used for the forecast, it is only an educated guess as to future values.

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B. THE Z-CHART

We will conclude this study unit with a short description of a particular type of chart which plots a time series, called a Z-Chart. It is basically a means of showing three sets of data relating to the performance of an organisation over time. The three sets of data are plotted on the same chart and should be kept up-to-date. The graphs are:

- a) The plot of the current data, be it monthly, quarterly or daily.
- b) The cumulative plot of the current data.
- c) The moving total plot of the data.

It is often used to keep senior management informed of business developments. As an example we will plot a Z-Chart for the sales of premium bonds in 19.5 using the data of the table below with the sales broken down into months. the table also shows the cumulative monthly sales and the moving annual totals. Note that the scale used for (a) is shown on the right of the chart and is twice that used for (b) and (c) so that the fluctuations in monthly sales show up more clearly. This is a device often used so that the chart is not too large.

Year	Month	Sales	Cumulative Sales	Moving Annual Total
19.5	Jan	150	150	1,240
	Feb	350	500	1,290
	Mar	300	800	1,460
	Apr	100	900	1,640
	May	150	1,050	1,670
	June	150	1,200	1,730
	July	120	1,320	1,830
	Aug	120	1,440	1,890
	Sept	100	1,540	1,940
	Oct	300	1,840	1,990
	Nov	400	2,240	2,140
	Dec	200	2,440	2,340

Figure 14.1

These totals are presented in Figure 10.2. It is called a Z-Chart because the position of the three graphs on the chart makes it look like the letter Z.

This is a useful chart because management can see at a glance how production is progressing from one month to the next. It is also possible to compare the current year's performance with a set target or with the same periods in previous years.

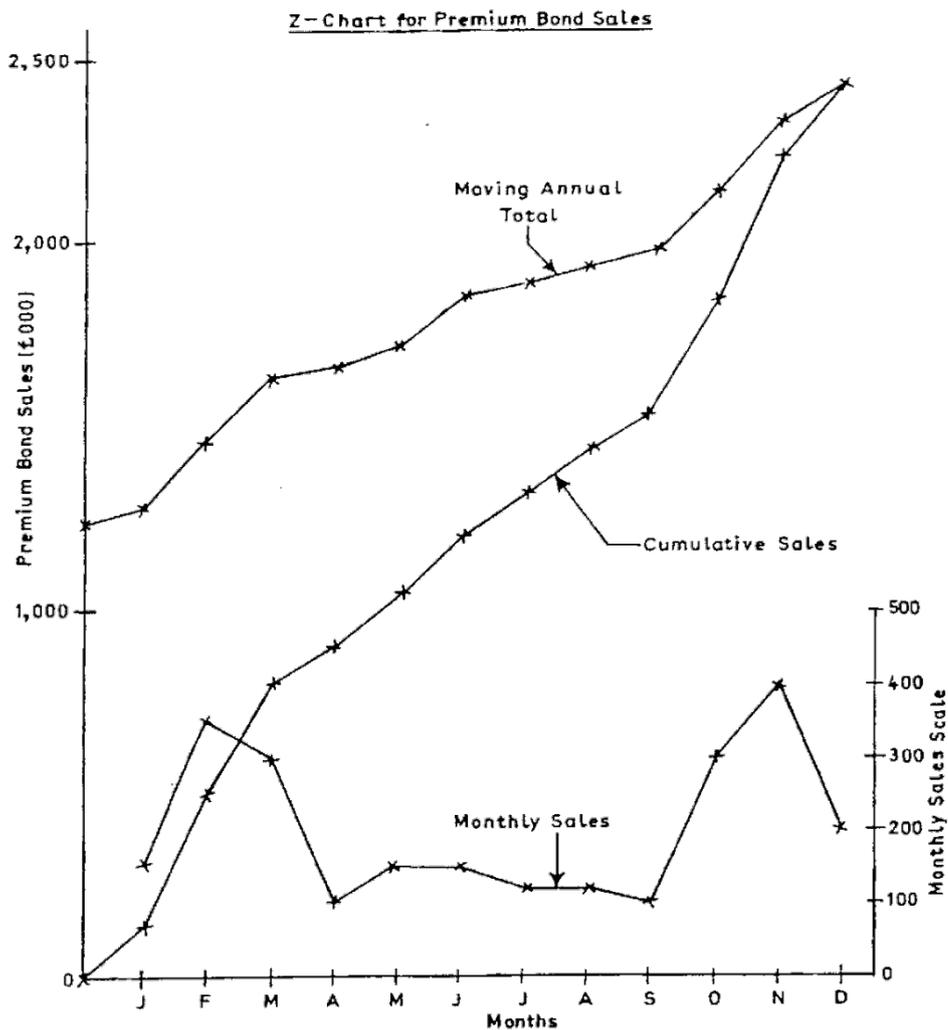


Figure 14.2

C. SUMMARY

In this study unit and the previous one we discussed the main models used to analyse time series. We began by identifying the various factors into which a time series may be divided in order to use these models, and went on to show how to separate a time series into these constituent factors. This is an important subject and you should particularly note the following points:

- Set out all calculations systematically in tables.
- The layout of the table used for calculation of centred moving averages is very important for all models.

You must learn thoroughly the method of calculating and adjusting seasonal variations for all models.

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STUDY UNIT 15

Network Analysis

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A. INTRODUCTION

Network analysis looks at the way large projects such as construction projects, computerisation projects or managerial projects are planned and executed. The objective of network analysis is to set out a framework or plan such that the project is carried out and completed efficiently.

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B. DRAWING A NETWORK DIAGRAM

Supposing a project is to build a house. Then in network analysis each stage of the process would be itemized and the time taken to do each part estimated. The logical sequence of events would also be thought through and written down. A network or diagram of the events would then be drawn up and the critical path identified.

The critical path

If all the various ways (paths) through a network are identified and timed, the path with the longest time is known as the critical path. If an activity on the critical path is delayed, then the entire project will be delayed. Activities on the critical path must finish on time if the project is to finish on time.

A network diagram essentially illustrates graphically the most efficient way of carrying out a particular project.

Once the basic concepts of drawing the network have been understood you can then complete the analysis of the network. This involves adding the duration or time that each activity takes and establishing the critical path through the network. It is often also necessary to find the earliest and latest event times.

Usually in exams you are given what is called a precedence table. The following table is an example of a typical precedence table. The activities could relate to any project. In this case let us assume it relates to building a garage.

Table 15.1

Activity	Preceding activity	Time for each activity (duration)
A. Buy bricks	- (None)	2
B. clear site	-	3
C. sort bricks	A	4
D. lay bricks	B	1
E. put on roof	CD	7

Usually in an exam you are asked to

- Draw the network based on the table.
- Find the critical path. This is the longest path through the network.

The Rules

There are various rules associated with drawing a network. These are as follows:

Rule 1

Each network starts and ends at just one point or node. The starting points are called nodes and the arrows joining the nodes are called activities. The diagram below represents a very simple network which would be written as follows.

Table 15.2

Activity	Preceding activity	Duration
A	-	2 weeks
B	A	3 weeks
C	B	2 weeks

A starts at the beginning, B comes after A and C comes after B. The figures in brackets represents the duration of each activity. The time to finish this project would be 7 weeks. (2+3+2). This network is the simplest type.

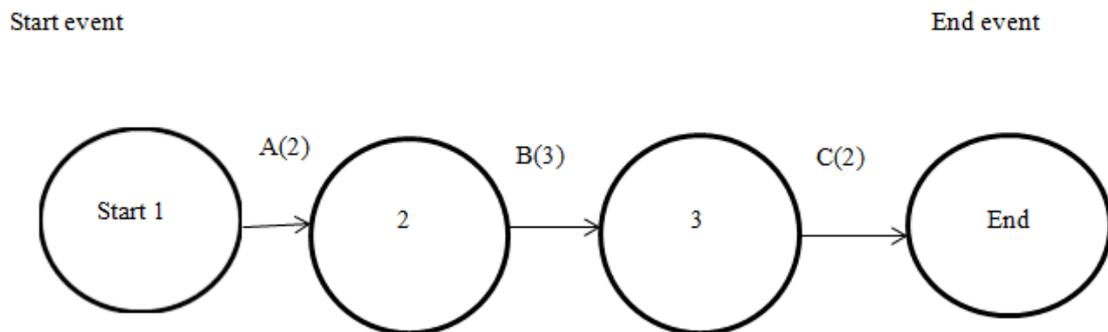


Figure 15.1

Rule 2

When drawing networks, the activities, arrows, cannot cross over each other.

Rule 3

The final rule is that two activities cannot start and end at the same node.

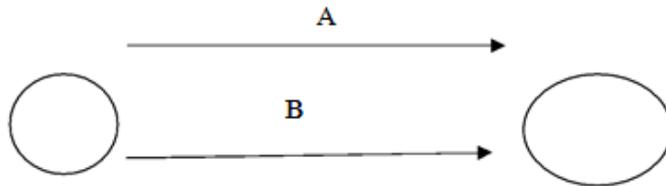


Figure 15.2

This is not allowed. If this occurs then you should use a dummy variable.

What is a dummy variable and when do you use one?

If the details of the network lead to two activities starting and finishing at the same nodes, it is avoided using a Dummy variable. This is an activity that just links nodes together. No duration attaches to it. For example:

Table 15.3

Activity	Preceding activity	Duration of activity
A	-	2
B	-	4
C	AB	5

In the above example, activities A & B both start at the beginning and must finish together before C can start. The network for this should be drawn as follows: the nodes are numbers for clarity so in the example below it would not matter if the 1 and 2 nodes were reversed. What is important with network analysis is that the network is logical and adheres to the rules of network drawing. Remember to always start and finish at one point only and the network should progress from left to right.

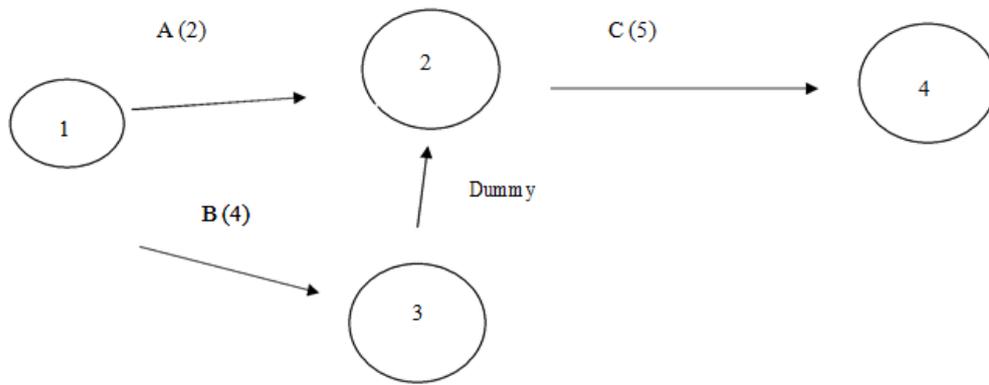


Figure 15.3

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C. EXAMPLES

We are now ready to do the example at the start of this section:

Table 15.4

Activity	Preceding activity	Time for each activity (duration)
A	- (None)	2
B	-	3
C	A	4
D	B	1
E	CD	7

- Draw the network based on the table.
- Find the critical path. This is the longest path through the network. $A+C+E = 2+4+7=13$

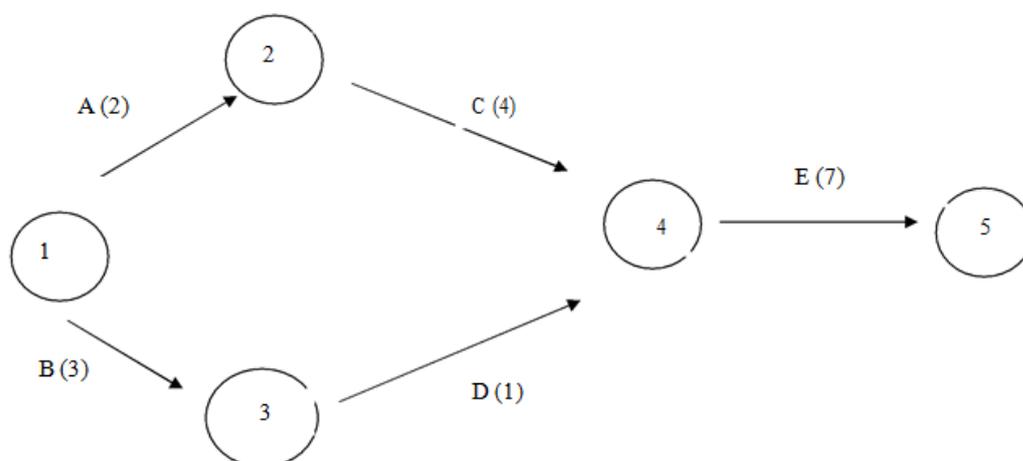


Figure 15.4

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STUDY UNIT 16

Linear Programming

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A. THE GRAPHICAL METHOD

The graphical method of linear programming is used for problems involving two products.

Formulating the problem

Let us suppose that WX manufactures two products, A and B. Both products pass through two production departments, mixing and shaping. The organisation's objective is to maximise contribution to fixed costs.

Product A is sold for RWF1.50 whereas product B is priced at RWF2.00. There is unlimited demand for product A but demand for B is limited to 13,000 units per annum. The machine hours available in each department are restricted to 2,400 per annum. Other relevant data are as follows.

Machine hours required	Mixing	Shaping
	Hrs	Hrs
Product A	0.06	0.04
Product B	0.08	0.12

Variable cost per unit	RWF
Product A	1.30
Product B	1.70

Before we work through the steps involved in solving this constraints problem using the graphical approach to linear programming, it is worth reading the CIMA Official Terminology definition of linear programming to get a glimpse of what we will be doing.

Linear programming is 'The use of a series of linear equations to construct a mathematical model. The objective is to obtain an optimal solution to a complex operational problem, which may involve the production of a number of products in an environment in which there are many constraints'.

What are the constraints in the situation facing WX?

- (i) Machine hours in each department
- (ii) Labour hours in each department
- (iii) Sales demand for product B
- (iv) Selling price of product A

- A. and (iii)
- B. only
- C. and (iv)
- D. (i), (ii) and (iii)

The correct answer is A. There is no restriction on the availability of labour hours. Selling price cannot be a constraint.

The steps in the **graphical method** are as follows.

- Define variables.
- Establish objective function.
- Establish constraints.
- Draw a graph of the constraints.
- Establish the feasible region.
- Determine the optimal product mix.

Let's start solving WX's problem.

Step 1

Define variables

What are the quantities that WX can vary? Obviously not the number of machine hours or the demand for product B. The only things which it can vary are the number of units of each type of product produced. It is those numbers which the company has to determine in such a way as to obtain the maximum possible profit. Our variables (which are usually products being produced) will therefore be as follows.

Let x = number of units of product A produced.

Let y = number of units of product B produced.

Step 2

Establish objective function

The **objective function** is a quantified statement of the aim of a resource allocation decision.

We now need to introduce the question of contribution or profit. We know that the contribution on each type of product is as follows.

	RWF per unit
Product A	$\text{RWF}(1.50 - 1.30) = 0.20$
Product B	$\text{RWF}(2.00 - 1.70) = 0.30$

The objective of the company is to maximise contribution and so the objective function to be maximised is as follows.

$$\text{Contribution (C)} = 0.2x + 0.3y$$

Step 3

Establish constraints

A **constraint** is 'An activity, resource or policy that limits the ability to achieve objectives'.

The value of the objective function (the maximum contribution achievable from producing products A and B) is limited by the constraints facing WX, however. To

incorporate this into the problem we need to translate the constraints into inequalities involving the variables defined in Step 1. An inequality is an equation taking the form 'greater than or equal to' or 'less than or equal to'.

- a) Consider the mixing department machine hours constraint.
- (i) Each unit of product A requires 0.06 hours of machine time. Producing five units therefore requires 5×0.06 hours of machine time and, more generally, producing x units will require $0.06x$ hours.
 - (ii) Likewise producing y units of product B will require $0.08y$ hours.
 - (iii) The total machine hours needed in the mixing department to make x units of product A and y units of product B is $0.06x + 0.08y$.
 - (iv) We know that this cannot be greater than 2,400 hours and so we arrive at the following inequality.

$$0.06x + 0.08y \leq 2,400$$

How can the constraint facing the shaping department be written as an inequality?

- A. $0.4x + 0.012y \leq 2,400$
- B. $0.04x + 0.12y \leq 2,400$
- C. $0.4x + 0.012y \leq 2,400$
- D. $0.04x + 0.12y \leq 2,400$

The correct answer is B. The constraint has to be a 'less than equal to' inequality, because the amount of resource used ($0.04x + 0.12y$) has to be 'less than equal to' the amount available of 2,400 hours.

- b) The final inequality is easier to obtain. The number of units of product B produced and sold is y but this has to be less than or equal to 13,000. Our inequality is therefore as follows.

$$y \leq 13,000$$

- c) We also need to add non-negativity constraints ($x \geq 0$, $y \geq 0$) since negative numbers of products cannot be produced. (Linear programming is simply a mathematical tool and so there is nothing in this method which guarantees that the answer will 'make sense'. An unprofitable product may produce an answer which is negative. This is mathematically correct but nonsense in operational terms. Always remember to include the non-negativity constraints. The examiner will not appreciate 'impossible' solutions.)

The problem has now been reduced to the following four inequalities and one equation.

Maximise contribution (C) = $0.2x + 0.3y$, subject to the following constraints:

$$\begin{array}{rcl} 0.06x + 0.08y & \leq & 2,400 \\ 0.04x + 0.12y & \leq & 2,400 \\ 0 \leq y & \leq & 13,000 \\ 0 & \leq & x \end{array}$$

Question

An organisation makes two products, X and Y. Product X has a contribution of RWF124 per unit and product Y RWF80 per unit. Both products pass through two departments for processing and the times in minutes per unit are as follows.

	Product X	Product Y
Department 1	150	90
Department 2	100	120

Currently there is a maximum of 225 hours per week available in department 1 and 200 hours in department 2. The organisation can sell all it can produce of X but EACEAC quotas restrict the sale of Y to a maximum of 75 units per week. The organisation, which wishes to maximise contribution, currently makes and sells 30 units of X and 75 units of Y per week.

Required

Assume x and y are the number of units of X and Y produced per week. Formulate a linear programming model of this problem, filling in the blanks in (a) and (b) below.

a) The objective function is to maximise weekly contribution, given by $C = \dots\dots\dots$

b) The constraints are:

Department 1 EAC quota

Department 2 Non-negativity

Answer

a) The objective function is to maximise weekly contribution, given by $C = 124x + 80y$.

b) The constraints are:

Department 1 $150x + 90y \leq 225 \times 60$ minutes

Department 2 $100x + 120y \leq 200 \times 60$ minutes

EAC quota $y \leq 75$

Non-negativity $x, y \geq 0$

These constraints can be simplified to:

Department 1 $15x + 9y \leq 1,350$

Department 2 $10x + 12y \leq 1,200$

EAC quota $y \leq 75$

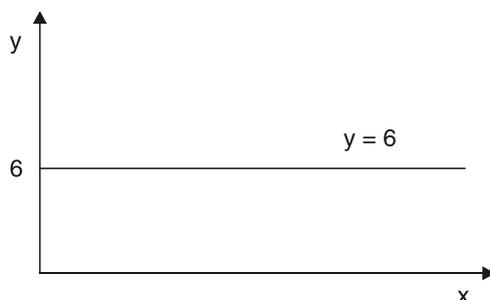
Non-negativity $x, y \geq 0$

Graphing the problem

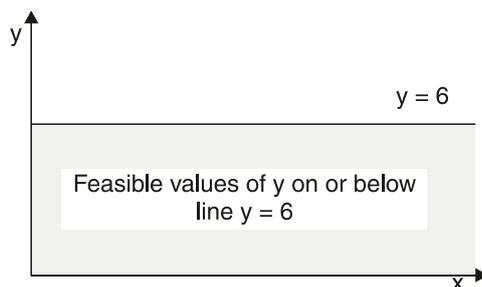
A graphical solution is only possible when there are two variables in the problem. One variable is represented by the x axis of the graph and one by the y axis. Since non-negative values are not usually allowed, the graph shows only zero and positive values of x and y.

Graphing equations and constraints

A linear equation with one or two variables is shown as a straight line on a graph. Thus $y = 6$ would be shown as follows.



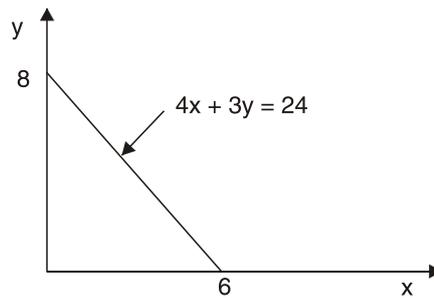
If the problem included a constraint that y could not exceed 6, the inequality $y \leq 6$ would be represented by the shaded area of the graph below.



The equation $4x + 3y = 24$ is also a straight line on a graph. To draw any straight line, we need only to plot two points and join them up. The easiest points to plot are the following.

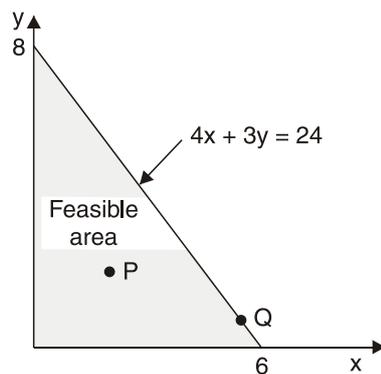
- $x = 0$ (in this example, if $x = 0$, $3y = 24$, $y = 8$)
- $y = 0$ (in this example, if $y = 0$, $4x = 24$, $x = 6$)

By plotting the points, $(0, 8)$ and $(6, 0)$ on a graph, and joining them up, we have the line for $4x + 3y = 24$.



Any combination of values for x and y on the line satisfies the equation. Thus at a point where $x = 3$ and $y = 4$, $4x + 3y = 24$. Similarly, at a point where $x = 4.5$ and $y = 2$, $4x + 3y = 24$.

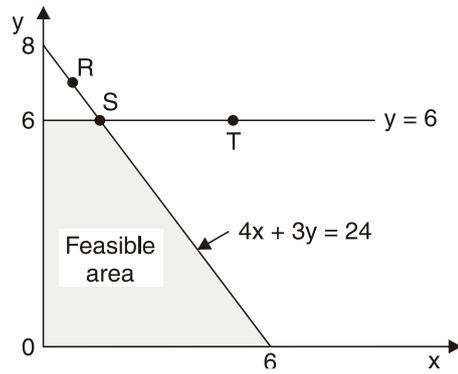
If we had a constraint $4x + 3y \leq 24$, any combined value of x and y within the shaded area below (on or below the line) would satisfy the constraint.



Consider point P which has coordinates of (2, 2). Here $4x + 3y = 14$, which is less than 24; and at point Q where $x = 5\frac{1}{2}$, $y = \frac{2}{3}$, $4x + 3y = 24$. Both P and Q lie within the feasible area or feasible region. A feasible area enclosed on all sides may also be called a feasible polygon.

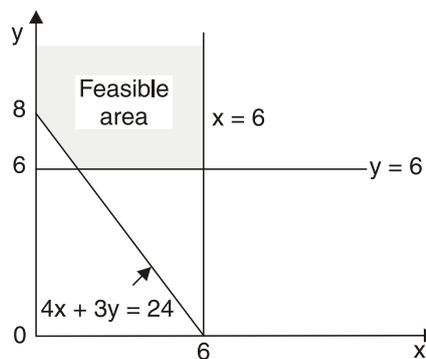
A feasible region is 'The area contained within all of the constraint lines shown on a graphical depiction of a linear programming problem. All feasible combinations of output are contained within or located on the boundaries of the feasible region'.

When there are several constraints, the feasible area of combinations of values of x and y must be an area where all the inequalities are satisfied. Thus, if $y \leq 6$ and $4x + 3y \leq 24$ the feasible area would be the shaded area in the following graph.



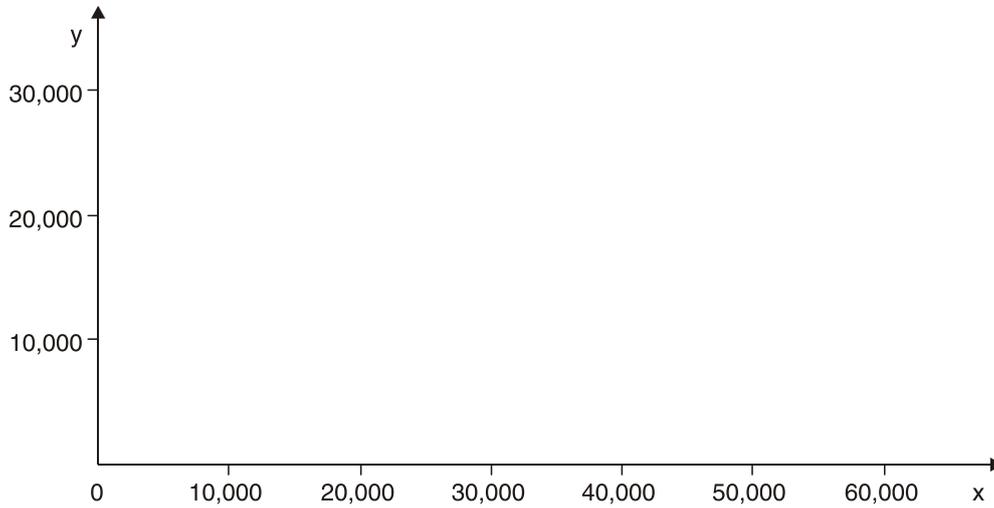
- (Point R ($x = 0.75$, $y = 7$) is not in the feasible area because although it satisfies the inequality $4x + 3y \leq 24$, it does not satisfy $y \leq 6$.
- Point T ($x = 5$, $y = 6$) is not in the feasible area, because although it satisfies the inequality $y \leq 6$, it does not satisfy $4x + 3y \leq 24$.
- Point S ($x = 1.5$, $y = 6$) satisfies both inequalities and lies just on the boundary of the feasible area since $y = 6$ exactly, and $4x + 3y = 24$. Point S is thus at the intersection of the two lines.

Similarly, if $y \geq 6$ and $4x + 3y \geq 24$ but x is ≤ 6 , the feasible area would be the shaded area in the graph below.



Question

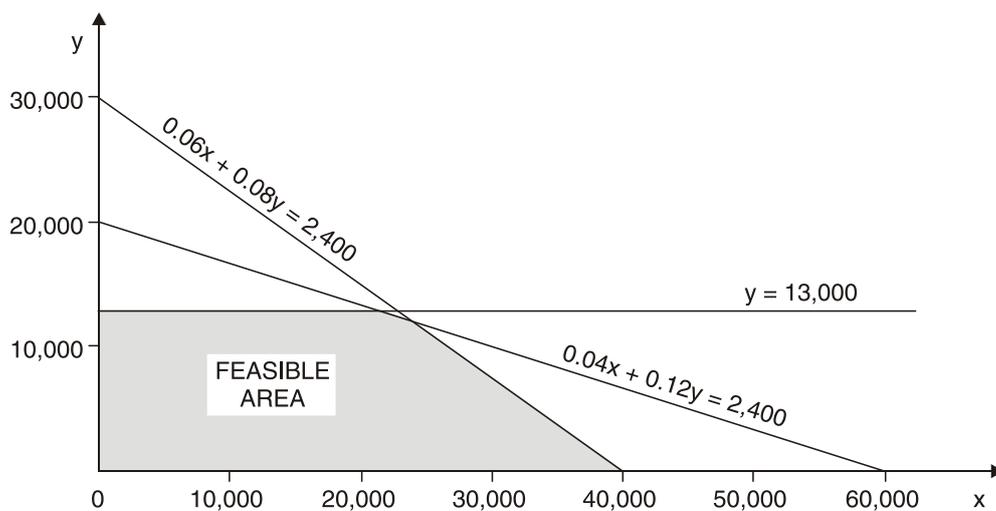
Draw the feasible region which arises from the constraints facing WX on the graph below.



Answer

If $0.06x + 0.08y = 2,400$, then if $x = 0$, $y = 30,000$ and if $y = 0$, $x = 40,000$.

If $0.04x + 0.12y = 2,400$, then if $x = 0$, $y = 20,000$ and if $y = 0$, $x = 60,000$.



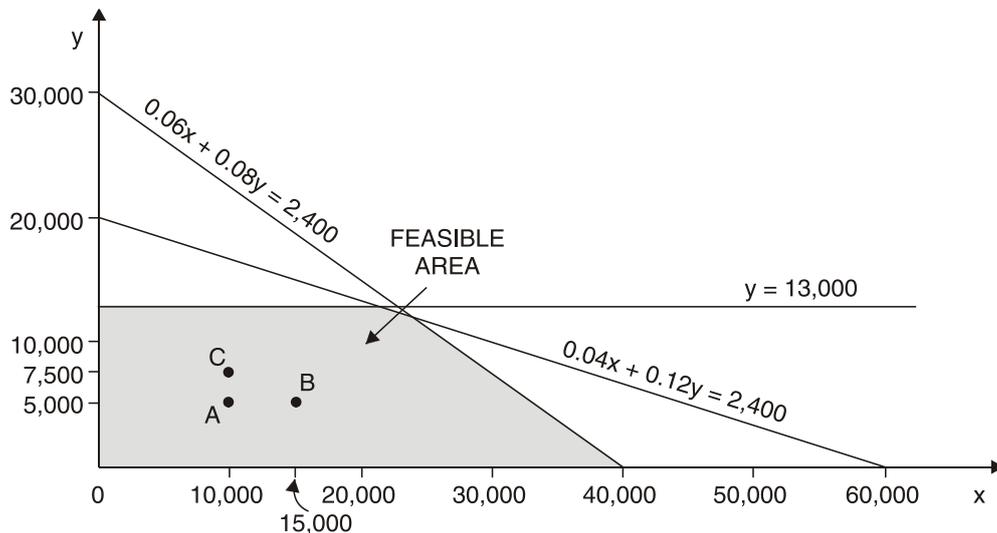
Finding the optimum allocation of resources

The **optimal solution** can be found by 'sliding the iso-contribution (or profit) line out'.

Having found the feasible region (which includes all the possible solutions to the problem) we need to find which of these possible solutions is 'best' or optimal in the sense that it yields the maximum possible contribution.

Look at the feasible region of the problem faced by WX (see the solution to the question above). Even in such a simple problem as this, there are a great many possible solution points within the feasible area. Even to write them all down would be a time-consuming process and also an unnecessary one, as we shall see.

Here is the graph of WX's problem.



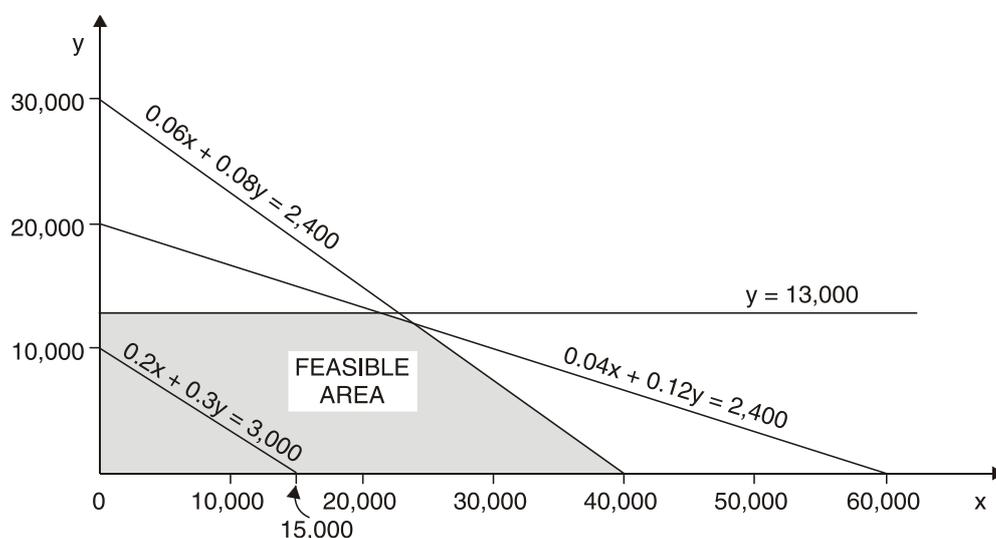
- Consider point A at which 10,000 units of product A and 5,000 units of product B are being manufactured. This will yield a contribution of $(10,000 \times \text{RWF}0.20) + (5,000 \times \text{RWF}0.30) = \text{RWF}3,500$.
- We would clearly get more contribution at point B, where the same number of units of product B are being produced but where the number of units of product A has increased by 5,000.
- We would also get more contribution at point C where the number of units of product A is the same but 2,500 more units of product B are being produced.

This argument suggests that the 'best' solution is going to be at a point on the edge of the feasible area rather than in the middle of it.

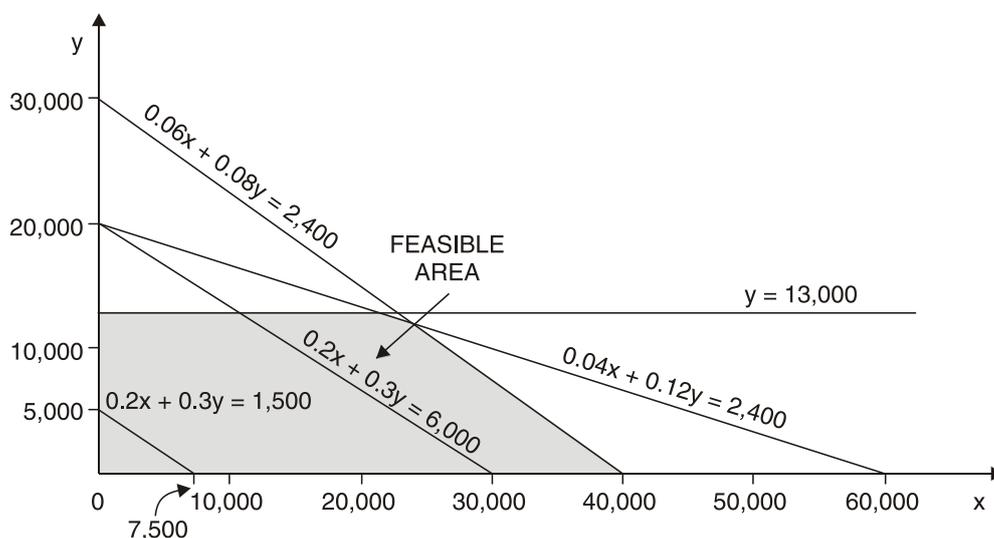
This still leaves us with quite a few points to look at but there is a way in which we can narrow down still further the likely points at which the best solution will be found. Suppose that WX wishes to earn contribution of RWF3,000. The company could sell the following combinations of the two products.

- a) 15,000 units of A, no B.
- b) No A, 10,000 units of B.
- c) A suitable mix of the two, such as 7,500 A and 5,000 B.

The possible combinations required to earn contribution of RWF3,000 could be shown by the straight line $0.2x + 0.3y = 3,000$.



Likewise for profits of RWF6,000 and RWF1,500, lines of $0.2x + 0.3y = 6,000$ and $0.2x + 0.3y = 1,500$ could be drawn showing the combination of the two products which would achieve contribution of RWF6,000 or RWF1,500.



The contribution lines are all parallel. (They are called **iso-contribution** lines, 'iso' meaning equal.) A similar line drawn for any other total contribution would also be parallel to the three lines shown here. Bigger contribution is shown by lines further from the origin ($0.2x + 0.3y = 6,000$), smaller contribution by lines closer to the origin ($0.2x + 0.3y = 1,500$). As WX tries to increase possible contribution, we need to 'slide' any contribution line outwards from the origin, while always keeping it parallel to the other contribution lines.

As we do this there will come a point at which, if we were to move the contribution line out any further, it would cease to lie in the feasible region. Greater contribution could not be achieved, because of the constraints. In our example concerning WX this will happen, as you should test for yourself, where the contribution line just passes through the intersection of $0.06x + 0.08y = 2,400$ and $0.04x + 0.12y = 2,400$ (at coordinates (24,000, 12,000)). The point (24,000, 12,000) will therefore give us the optimal allocation of resources (to produce 24,000 units of A and 12,000 units of B).

We can usefully summarise the graphical approach to linear programming as follows.

Step 1 Define variables.

Step 4 Graph the problem.

Step 2 Establish objective function.

Step 5 Define feasible area.

Step 3 Establish constraints.

Step 6 Determine optimal solution.

Example: the graphical solution with a twist

This example shows that it is not always necessarily easy to identify the decision variables in a problem.

DCC operates a small plant for the manufacture of two joint chemical products X and Y. The production of these chemicals requires two raw materials, A and B, which cost RWF5 and RWF8 per litre respectively. The maximum available supply per week is 2,700 litres of A and 2,000 litres of B.

The plant can operate using either of two processes, which have differing operating costs and raw materials requirements for the production of X and Y, as follows.

Process	Raw materials consumed		Output		Cost
	Litres per processing hour		Litres per hour		RWF per hour
	A	B	X	Y	
1	20	10	15	20	500
2	30	20	20	10	230

The plant can run for 120 hours per week in total, but for safety reasons, process 2 cannot be operated for more than 80 hours per week.

X sells for RWF18 per litre, Y for RWF24 per litre.

Formulate a linear programming model, and then solve it, to determine how the plant should be operated each week.

Solution

Step 1 Define variables

You might decide that there are two decision variables in the problem, the quantity of X and the quantity of Y to make each week. If so, begin by letting these be x and y respectively.

You might also readily recognise that the aim should be to maximise the total weekly contribution, and so the objective function should be expressed in terms of maximising the total contribution from X and Y.

The contribution per litre from X and Y cannot be calculated because the operating costs are expressed in terms of processing hours.

	<i>Process 1</i>			<i>Process 2</i>		
	RWF	perRWF	per	RWF	perRWF	per
	hour	hour		hour	hour	
Costs:						
Material A		100			150	
Material B		80			160	
Operating cost		<u>500</u>			<u>230</u>	
		680			540	
Revenue:						
X	(15 × RWF18)	270	(20	×360		
			RWF18)			
Y	(20 × RWF24)	<u>480</u>	(10	× <u>240</u>		
			RWF24)			
		<u>750</u>			<u>600</u>	
Contribution		<u><u>70</u></u>			<u><u>60</u></u>	

The decision variables should be processing hours in each process, rather than litres of X and Y. If we let the processing hours per week for process 1 be P1 and the processing hours per week for process 2 be P2 we can now formulate an objective function, and constraints, as follows.

Step 2 Establish objective function

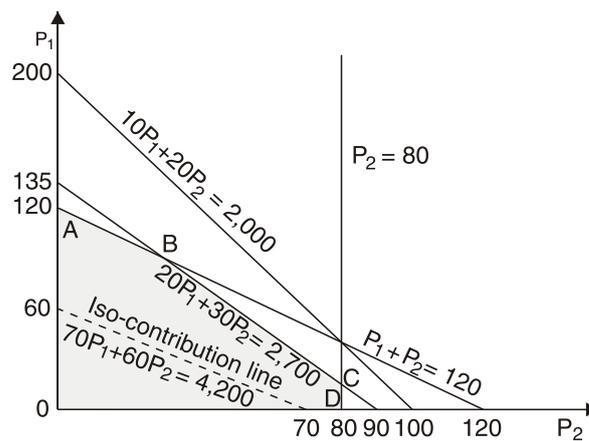
Maximise $70P_1 + 60P_2$ (total contribution) subject to the constraints below

Step 3 Establish constraints

$$\begin{aligned}
 20P_1 + 30P_2 &\leq 2,700 && \text{(material A supply)} \\
 10P_1 + 20P_2 &\leq 2,000 && \text{(material B supply)} \\
 P_2 &\leq 80 && \text{(maximum time for } P_2) \\
 P_1 + P_2 &\leq 120 && \text{(total maximum time)} \\
 P_1, P_2 &\geq 0 &&
 \end{aligned}$$

Step 4 Graph the problem

The graphical solution looks like this.



Step 5 Define feasible area

The material B constraint is not critical, and the feasible area for a solution is shown as ABCDO on the graph.

Step 6 Determine optimal solution

The optimal solution, determined using the iso-contribution line $70P_1 + 60P_2 = 4,200$, is at point A, where $P_1 = 120$ and $P_2 = 0$.

Production would be (120×15) 1,800 litres of X and (120×20) 2,400 litres of Y.

Total contribution would be $(120 \times \text{RWF}70) = \text{RWF}8,400$ per week.

Question

On 20 days of every month GS makes two products, the Crete and the Corfu. Production is carried out in three departments – tanning, plunging and watering. Relevant information is as follows.

	Crete	Corfu
Contribution per unit	RWF75	RWF50
Minutes in tanning department per unit	10	12
Minutes in plunging department per unit	15	10
Minutes in watering department per unit	6	15
Maximum monthly sales (due to government quota restrictions)	3,500	4,000

	Tanning	Plunging	Watering
Number of employees	7	10	5
Hours at work per day per employee	7	6	10
Number of idle hours per day per employee	0.5	1	0.25

Due to union restrictions, employees cannot be at work for longer than the hours detailed above.

Use the graphical method of linear programming to determine the optimum monthly production of Cretes and Corfus and the monthly contribution if GS's objective is to maximise contribution.

Answer

Calculate the number of productive hours worked in each department each month

Number of employees x number of productive hours worked each day x number of days each month.

$$\text{Tanning} = 7 \times (7 - 0.5) \times 20 = 910 \text{ hours}$$

$$\text{Plunging} = 10 \times (6 - 1) \times 20 = 1,000 \text{ hours}$$

$$\text{Watering} = 5 \times (10 - 0.25) \times 20 = 975 \text{ hours}$$

Step 1 Define variables

Let the number of Cretes produced each month = x and the number of Corfus produced each month = y .

Step 2 Establish objective function

The contribution is RWF75 per Crete and RWF50 per Corfu. The objective function is therefore maximise $C = 75x + 50y$ subject to the constraints below.

Step 3 Establish constraints

$$\text{Tanning} \quad x/6 + y/5 \leq 910$$

$$\text{Plunging} \quad x/4 + y/6 \leq 1,000$$

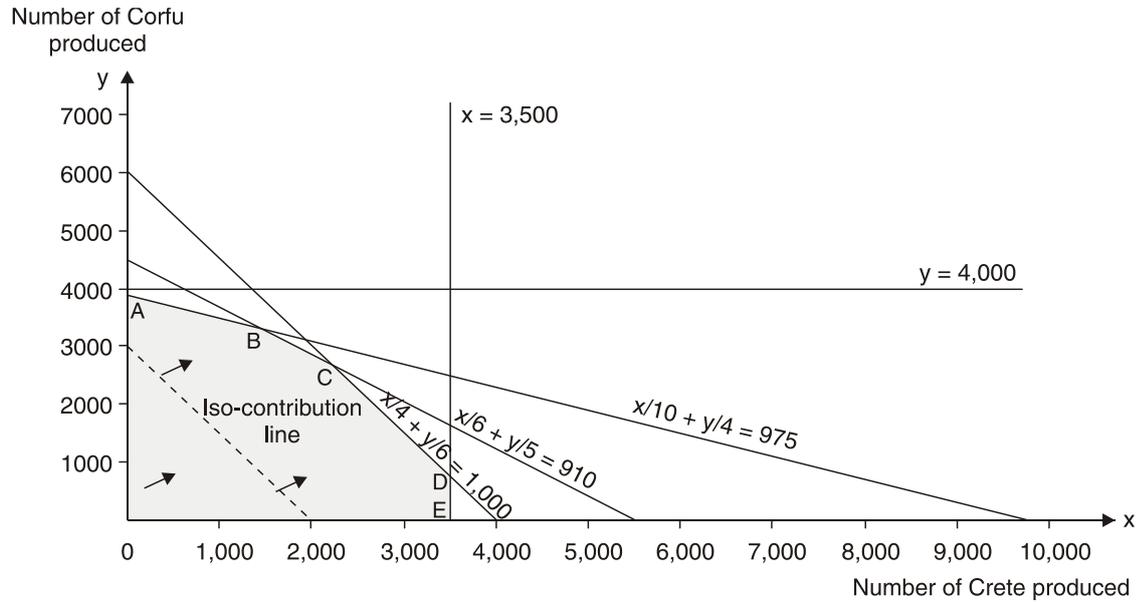
$$\text{Watering} \quad x/10 + y/4 \leq 975$$

$$\text{Monthly sales units} \quad x \leq 3,500, y \leq 4,000$$

$$\text{Non negativity} \quad x \geq 0, y \geq 0$$

Step 4 Graph the problem

The problem can be solved using the following graph which includes a sample contribution line $75x + 50y = 150,000$.



Step 5 Define the feasible area

The feasible region for a solution is OABCDE.

Step 6 Determine the optimal solution

Moving the sample contribution line across the feasible region it can be seen that the optimum solution is at any point along the line $x/4 + y/6 = 1,000$ between C and D (as the sample contribution line has the same gradient as the plunging constraint). The coordinates of point C are (2,175, 2,737.5) while those of point D are (3,500, 750).

The contribution from any of these solutions is $RWF((75 \times 3,500) + (50 \times 750)) = RWF300,000$ (using the coordinates of D).

B. THE GRAPHICAL METHOD USING SIMULTANEOUS EQUATIONS

Instead of a 'sliding the contribution line out' approach, simultaneous equations can be used to determine the optimal allocation of resources, as shown in the following example.

The optimal solution can also be found using **simultaneous equations**.

Example: using simultaneous equations

An organisation manufactures plastic-covered steel fencing in two qualities: standard and heavy gauge. Both products pass through the same processes involving steel forming and plastic bonding.

The standard gauge sells at RWF15 a roll and the heavy gauge at RWF20 a roll. There is an unlimited market for the standard gauge but outlets for the heavy gauge are limited to 13,000 rolls a year. The factory operations of each process are limited to 2,400 hours a year. Other relevant data is given below.

Variable costs per roll

	Direct Material	Direct Wages	Direct Expense
	<i>RWF</i>	<i>RWF</i>	<i>RWF</i>
Standard	5	7	1
Heavy	7	8	2

Processing hours per 100 roll

	Steel Forming	Plastic Bonding
	<i>Hours</i>	<i>Hours</i>
Standard	6	4
Heavy	8	12

Calculate the allocation of resources and hence the production mix which will maximise total contribution.

Solution

Step 1 Define variables

Let the number of rolls of standard gauge to be produced be x and the number of rolls of heavy gauge be y .

Step 2 Establish objective function

Standard gauge produces a contribution of RWF2 per roll (RWF15 – RWF(5 + 7 + 1)) and heavy gauge a contribution of RWF3 (RWF20 – RWF(7 + 8 + 2)).

Therefore the objective is to maximise contribution $(C) = 2x + 3y$ subject to the constraints below.

Step 3 Establish constraints

The constraints are as follows.

$$0.06x + 0.08y \leq 2,400 \quad (\text{steel forming hours})$$

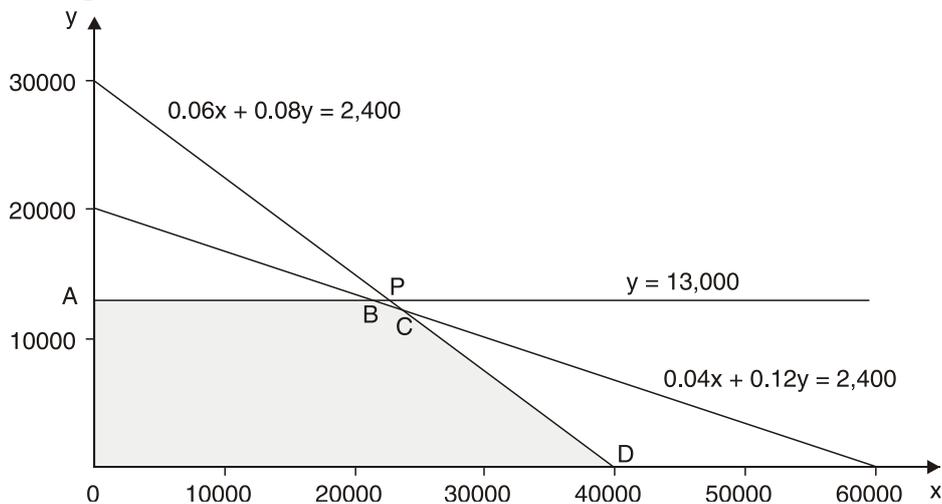
$$0.04x + 0.12y \leq 2,400 \quad (\text{plastic bonding hours})$$

$$y \leq 13,000 \quad (\text{demand for heavy gauge})$$

$$x, y \geq 0 \quad (\text{non-negativity})$$

Step 4 Graph problem

The graph of the problem can now be drawn.



Step 5 Define feasible area

The combinations of x and y that satisfy all three constraints are represented by the area OABCD.

Step 6 Determine optimal solution

Which combination will maximise contribution? Obviously, the more units of x and y, the bigger the contribution will be, and the optimal solution will be at point B, C or D. It will not be at A, since at A, $y = 13,000$ and $x = 0$, whereas at B, $y = 13,000$ (the same) and x is greater than zero.

Using simultaneous equations to calculate the value of x and y at each of points B, C and D, and then working out total contribution at each point from this, we can establish the contribution-maximising product mix.

Point B

$$\begin{aligned}y &= 13,000 \quad (1) \\0.04x + 0.12y &= 2,400 \quad (2) \\0.12y &= 1,560 \quad (3) \quad ((1) \times 0.12) \\0.04x &= 840 \quad (4) \quad ((2) - (3)) \\x &= 21,000 \quad (5)\end{aligned}$$

Total contribution = $(21,000 \times \text{RWF}2) + (13,000 \times \text{RWF}3) = \text{RWF}81,000$.

Point C

$$\begin{aligned}0.06x + 0.08y &= 2,400 \quad (1) \\0.04x + 0.12y &= 2,400 \quad (2) \\0.12x + 0.16y &= 4,800 \quad (3) \quad ((1) \times 2) \\0.12x + 0.36y &= 7,200 \quad (4) \quad ((2) \times 3) \\0.2y &= 2,400 \quad (5) \quad ((4) - (3)) \\y &= 12,000 \quad (6)\end{aligned}$$

$$0.06x + 960 = 2,400 \quad (7) \text{ (substitute in (1))}$$

$$x = 24,000 \quad (8)$$

Total contribution = $(24,000 \times \text{RWF}2) + (12,000 \times \text{RWF}3) = \text{RWF}84,000$.

Point D

Total contribution = $40,000 \times \text{RWF}2 = \text{RWF}80,000$.

Comparing B, C and D, we can see that contribution is maximised at C, by making 24,000 rolls of standard gauge and 12,000 rolls of heavy gauge, to earn a contribution of RWF84,000.

Slack and surplus

Slack occurs when maximum availability of a resource is not used. **Surplus** occurs when more than a minimum requirement is used.

If, at the optimal solution, the resource used equals the resource available there is no spare capacity of a resource and so there is no slack.

If a resource which has a maximum availability is not binding at the optimal solution, there will be slack.

In the example above, the optimal solution is $x = 24,000$, $y = 12,000$.

If we substitute these values into the inequalities representing the constraints, we can determine whether the constraints are binding or whether there is slack.

Steel forming hours: $(0.06 \times 24,000) + (0.08 \times 12,000) = 2,400 = \text{availability}$

Constraint is binding.

Plastic bonding hours: $(0.04 \times 24,000) + (0.12 \times 12,000) = 2,400 = \text{availability}$

Constraint is binding.

Demand: Demand of 12,000 \leq maximum demand of 13,000

There is slack.

Note that because we had already determined the optimal solution to be at the intersection of the steel forming hours and plastic bonding hours constraints, we knew that they were binding!

If a minimum quantity of a resource must be used and, at the optimal solution, more than that quantity is used, there is a surplus on the minimum requirement.

For example, suppose in a particular scenario a minimum of 8,000 grade A labour hours had to be worked in the production of products x and y , such that (say) $3x + 2y \geq 8,000$. If 10,000 hours are used to produce the optimal solution, there is a surplus of 2,000 hours.

We will be looking at this form of constraint in the next section.

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C. Sensitivity analysis

Once a graphical linear programming solution has been found, it should be possible to provide further information by interpreting the graph more fully to see what would happen if certain values in the scenario were to change.

- a) What if the contribution from one product was RWF1 lower than expected?
- b) What if the sales price of another product was raised by RWF2?
- c) What would happen if less or more of a limiting factor were available, such as material?

Sensitivity analysis with linear programming can be carried out in one of two ways.

- a) By considering the value of each limiting factor or binding resource constraint
- b) By considering sale prices (or the contribution per unit)

Limiting factor sensitivity analysis

We use the shadow price to carry out sensitivity analysis on the availability of a limiting factor.

Shadow prices

The shadow price of a resource which is a limiting factor on production is the amount by which total contribution would fall if the organisation were deprived of one unit of the resource. The shadow price also indicates the amount by which total contribution would rise if the organisation were able to obtain one extra unit of the resource, provided that the resource remains an effective constraint on production and provided also that the extra unit of resource can be obtained at its normal variable cost.

Question

Choose the correct words from those highlighted.

A shadow price is the **increase/decrease in contribution/revenue** created by the availability of an extra unit of a **resource/limiting** resource at **its original cost/a premium price**.

Answer

The correct answer is: A shadow price is the increase in contribution created by the availability of an extra unit of a limiting resource at its original cost.

So in terms of linear programming, the shadow price is the extra contribution or profit that may be earned by relaxing by one unit a binding resource constraint.

Suppose the availability of materials is a binding constraint. If one extra kilogram becomes available so that an alternative production mix becomes optimal, with a resulting increase over the original production mix contribution of RWF2, the shadow price of a kilogram of material is RWF2.

Note, however, that this increase in contribution of RWF2 per extra kilogram of material made available is calculated on the assumption that the extra kilogram would cost the normal variable amount.

Note the following points.

- a) The shadow price therefore represents the maximum premium above the basic rate that an organisation should be willing to pay for one extra unit of a resource.
- b) Since shadow prices indicate the effect of a one unit change in a constraint, they provide a measure of the sensitivity of the result.
- c) The shadow price of a constraint that is not binding at the optimal solution is zero.
- d) Shadow prices are only valid for a small range before the constraint becomes non-binding or different resources become critical.

Depending on the resource in question, shadow prices enable management to make **better informed decisions** about the payment of overtime premiums, bonuses, premiums on small orders of raw materials and so on.

Calculating shadow prices

In the earlier example of WX, the availability of time in both departments are limiting factors because both are used up fully in the optimal product mix. Let us therefore calculate the effect if one extra hour of shaping department machine time was made available so that 2,401 hours were available.

The new optimal product mix would be at the intersection of the two constraint lines $0.06x + 0.08y = 2,400$ and $0.04x + 0.12y = 2,401$.

Solution by simultaneous equations gives $x = 23,980$ and $y = 12,015$.

(You should solve the problem yourself if you are doubtful about the derivation of the solution.)

Product	Units	Contribution per unit	Total Contribution
		RWF	RWF
A	23,980	0.20	4,796.0
B	12,015	0.30	<u>3,604.5</u>
			8,400.5

Contribution in original problem

$$((24,000 \times \text{RWF}0.20) + (12,000 \times \text{RWF}0.30)) \quad \underline{8,400.0}$$

$$\text{Increase in contribution from one extra hour of shaping time} \quad \underline{\underline{0.5}}$$

The **shadow price of an hour of machining time in the shaping department is therefore RWF0.50.**

The shadow price of a limiting factor also shows by how much contribution would fall if the availability of a limiting resource fell by one unit. The shadow price (also called dual price) of an hour of machine time in the shaping department would again be calculated as RWF0.50. This is the opportunity cost of deciding to put an hour of shaping department time to an alternative use.

We can now make the following points.

- The management of WX should be prepared to pay up to RWF0.50 extra per hour (ie RWF0.50 over and above the normal price) of shaping department machine time to obtain more machine hours.

- b) This value of machine time only applies as long as shaping machine time is a limiting factor. If more and more machine hours become available, there will eventually be so much machine time that it is no longer a limiting factor.

Question

What is the shadow price of one hour of machine time in the mixing department?

- A. RWF3
- B. RWF7
- C. RWF10.50
- D. RWF1,193

Answer

The correct answer is A.

If we assume one less hour of machine time in the mixing department is available, the new optimal solution is at the intersection of $0.06x + 0.08y = 2,399$ and $0.04x + 0.12y = 2,400$

Solution by simultaneous equations gives $x = 23,970$, $y = 12,010$

		Contribution per unit	Total Contribution
Product	Units	RWF	RWF
A	23,970	0.20	4,794
B	12,010	0.30	<u>3,603</u>
			8,397
Contribution in original problem			<u>8,400</u>
Reduction in contribution			<u><u>3</u></u>

Therefore shadow price of one hour of machine time in the mixing department is RWF3.

Ranges for limiting factors

We can calculate how many hours will be available before machine time in the shaping department ceases to be a limiting factor.

As more hours become available the constraint line moves out away from the origin. It ceases to be a limiting factor when it passes through the intersection of the sales constraint and the mixing department machine time constraint which is at the point (22,667, 13,000).

So, if $x = 22,667$ and $y = 13,000$, our new constraint would be $0.04x + 0.12y = H$ (hours) where $H = (0.04 \times 22,667) + (0.12 \times 13,000) = 2,466.68$ hours.

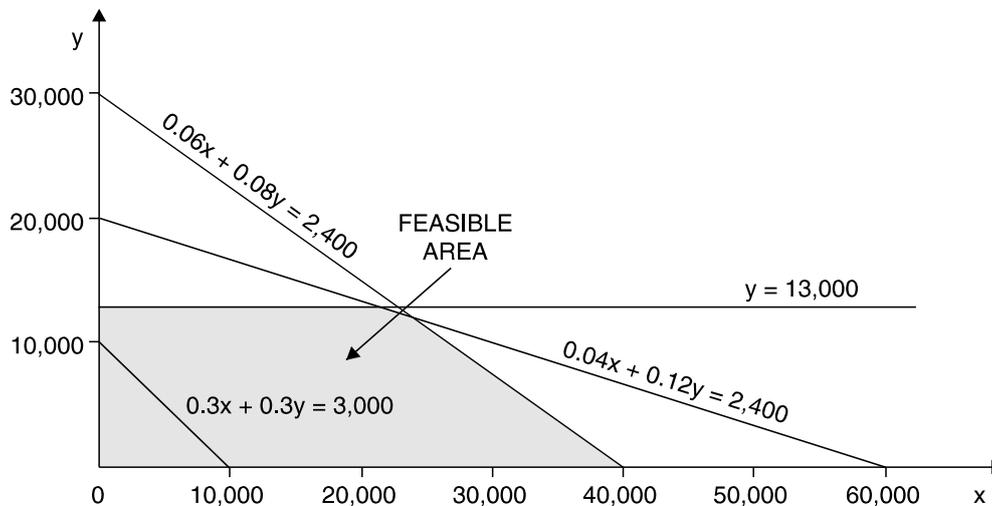
The shadow price of shaping department machine time is therefore RWF0.50 but only up to a maximum supply of 2,466.68 hours (that is 66.68 hours more than the original 2,400 hours). Extra availability of machine time above 2,466.68 hours would not have any use, and the two limiting factors would become sales demand for product B and machine time in the mixing department.

Sales price sensitivity analysis

Sales price sensitivity analysis is carried out by changing the slope of the 'iso-contribution' line.

The optimal solution in our WX example was to make 24,000 units of product A and 12,000 units of product B. Would this solution change if the unit sales price of A increased by 10RWF?

The contribution would increase to $0.3x + 0.3y$ (in place of $0.2x + 0.3y$). The iso-contribution lines would now have a steeper slope than previously, parallel (for example) to $0.3x + 0.3y = 3,000$.



If you were to place a ruler along the iso-contribution line and move it away from the origin as usual, you would find its **last point within the feasible region** was the point (40,000, 0).

Therefore if the sales price of A is raised by 10p, WX's contribution-maximising product mix would be to produce 40,000 units of A and none of B.

Example: sensitivity analysis

SW makes two products, X and Y, which each earn a contribution of RWF8 per unit. Each unit of X requires four labour hours and three machine hours. Each unit of Y requires three labour hours and five machine hours.

Total weekly capacity is 1,200 labour hours and 1,725 machine hours. There is a standing weekly order for 100 units of X which must be met. In addition, for technical reasons, it is necessary to produce at least twice as many units of Y as units of X.

- a) Determine the contribution-maximising production plan each week.
- b) Calculate the shadow price of the following.
 - (i) Machine hours
 - (ii) Labour hours
 - (iii) The minimum weekly demand for X of 100 units

Solution (a): production plan

The linear programming problem may be formulated as follows.

Step 1 Define variables

Let x = number of units of X produced and y = number of units of Y produced.

Step 2 Establish objective function

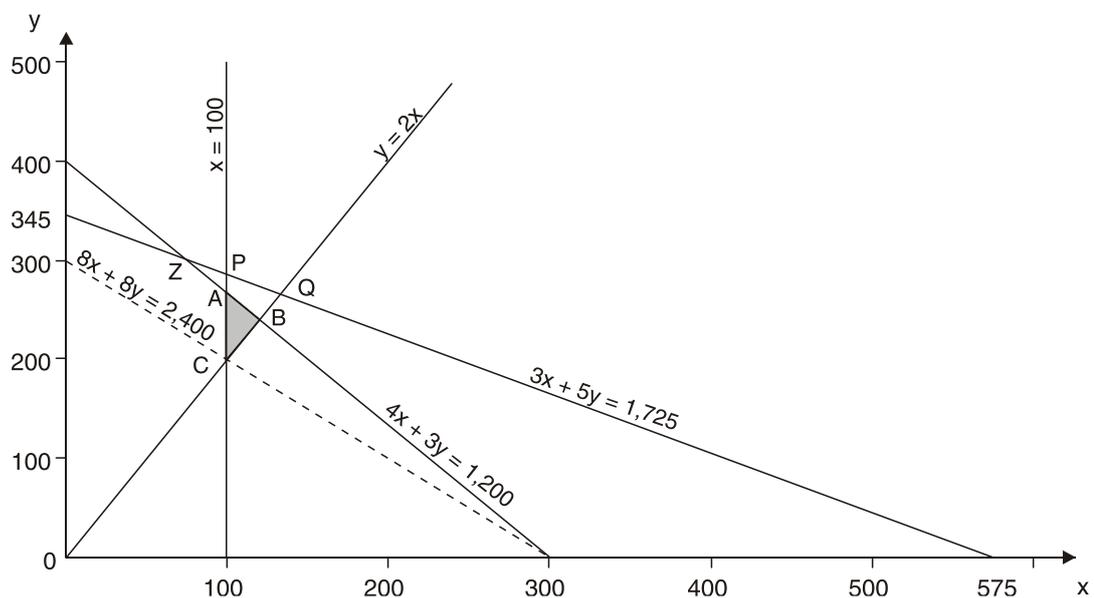
Maximise contribution (c) = $8x + 8y$ subject to the constraints below.

Step 3 Establish constraints

$$\begin{aligned} 4x + 3y &\leq 1,200 && \text{(labour hours)} \\ 3x + 5y &\leq 1,725 && \text{(machine hours)} \\ x &\geq 100 && \text{(minimum demand)} \\ y &\geq 2x && \text{(technical constraint)} \\ y &\geq 0 && \text{(non-negativity)} \end{aligned}$$

Step 4 Graph the problem

The graph of this problem would be drawn as follows, using $8x + 8y = 2,400$ as an iso-contribution line.



Step 5 Establish feasible polygon

The feasible polygon is ABC. Using the slope of the iso-contribution line, we can measure that the contribution-maximising point is point A.

Step 6 Determine optimal solution

At point A, the effective constraints are $x = 100$ and $4x + 3y = 1,200$.

$$\therefore \text{If } x = 100, (4 \times 100) + 3y = 1,200$$

$$\therefore 3y = 1,200 - 400 \text{ and so } y = 266\frac{2}{3}$$

It is important to be aware that in linear programming, the optimal solution is likely to give values to the decision variables which are in fractions of a unit. In this example, contribution will be maximised by making $266\frac{2}{3}$ units of Y.

	Contribution
	RWF
Make 100 units of X	800.00
$266\frac{2}{3}$ units of Y	<u>2,133.33</u>
Total weekly contribution	<u><u>2,933.33</u></u>

Solution (b): sensitivity analysis

- (i) Machine hours are not fully utilised in the optimal solution. 100 units of X and $266\frac{2}{3}$ units of Y need $(300 + 1,333.33) = 1,633.33$ machine hours, leaving 91.67 machine hours unused. Machine hours, not being an effective constraint in the optimal solution, have a shadow price of RWF0. Obtaining one extra machine hour would add nothing to the contribution.

- (ii) The shadow price of labour hours would be obtained by calculating the total weekly contribution if the labour hours constraint were 1,201 hours. It should be possible to see fairly easily that the new optimal solution would be where $x = 100$ and $4x + 3y = 1,201$. Therefore $x = 100$, $y = 267$ and total weekly contribution would be $(100 + 267) \times \text{RWF}8 = \text{RWF}2,936$.

Since contribution with 1,200 labour hours as the constraint was $\text{RWF}2,933.33$, the shadow price of labour hours is $\text{RWF}(2,936 - 2,933.33) = \text{RWF}2.67$ per hour. This is the amount by which total contribution would rise if one extra labour hour per week were made available.

Note that there is a limitation to the number of extra labour hours that could be used to earn extra contribution. As more and more labour hours are added, the constraint line will move further and further away from the origin. For example if we added 800 labour hours capacity each week, the constraint $4x + 3y \leq (1,200 + 800)$ (ie $4x + 3y \leq 2,000$) would be so much further away from the origin that it would no longer be an effective constraint. Machine hours would now help to impose limitations on production, and the profit-maximising output would be at point P on the graph.

Labour hours could only be added to earn more contribution up to point P, after which they would cease to be an effective constraint. At point P, $x = 100$ and $3x + 5y = 1,725$. Therefore $y = 285$.

The labour hours required to make 100 units of X and 285 units of Y are $(4 \times 100) + (3 \times 285) = 1,255$ hours, which is 55 hours more than the initial constraint limit.

Total contribution at point P = $(100 + 285) \times \text{RWF}8 = \text{RWF}3,080$. Since total contribution at point A, where labour hours were limited to 1,200 hours, was $\text{RWF}2,933.33$, the extra contribution from the 55 extra labour hours would be $\text{RWF}(3,080 - 2,933.33)/55 = \text{RWF}2.67$ per hour (as calculated previously).

Thus, the shadow price of labour hours is $\text{RWF}2.67$ per hour, for a maximum of 55 extra hours per week, after which additional labour hours would add nothing to the weekly contribution.

- (iii) The shadow price of the minimum weekly demand for X may be obtained by calculating the weekly contribution if the minimum demand is reduced by one unit to 99, so that $x \geq 99$, given no change in the other original constraints in the problem.

The new optimal solution would occur where $x = 99$ and $4x + 3y = 1,200$. Therefore $y = 268$.

Total contribution per week when $x = 99$ and $y = 268$ is $(99 + 268) \times \text{RWF}8 = \text{RWF}2,936$. Since the contribution when $x \geq 100$ was $\text{RWF}2,933.33$, the shadow price of the minimum demand for X is $\text{RWF}(2,936 - 2,933.33) = \text{RWF}2.67$ per unit. In other words, by reducing the minimum demand for X, the weekly contribution can be raised by $\text{RWF}2.67$ for each unit by which the minimum demand is reduced below 100 per week.

As with the constraint on labour hours, this shadow price is only applicable up to a certain amount. If you refer back to the graph of the problem, you should be able to see that if the minimum constraint on X is reduced beyond point Z, it will cease to be an effective constraint in the optimal solution, because at point Z the machine hours limitation will begin to apply.

Question

By how many units per week can the minimum demand be reduced before the shadow price of $\text{RWF}2.67$ per unit referred to above ceases to apply?

- A. 300 units
- B. 100 units
- C. 75 units
- D. 25 units

Answer

The correct answer is D.

At point Z: $4x + 3y = 1,200$ (1)

$3x + 5y = 1,725$ (2)

Multiply (1) by 3 $12x + 9y = 3,600$ (3)

Multiply (2) by 4 $12x + 20y = 6,900$ (4)

Subtract (3) from (4) $11y = 3,300$

$y = 300$

Substituting in (1) $4x + 900 = 1,200$

$4x = 300$

$x = 75$

The shadow price of the minimum demand for X is RWF2.67 per unit demanded, but only up to a total reduction in the minimum demand of $(100 - 75) = 25$ units per week.

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D. THE PRINCIPLES OF THE SIMPLEX METHOD

The **simplex method** is a method of solving linear programming problems with two or more decision variables.

The formulation of the problem using the **simplex method** is similar to that required when the graphical method is used but **slack variables** must be incorporated into the constraints and the objective function.

General points about the simplex method

A **slack variable** represents the amount of a constraint that is unused.

In any feasible solution, if a problem involves n constraints and m variables (decision plus slack), n variables will have a positive value and $(m-n)$ variables will have a value of zero.

Feasible solutions to a problem are shown in a **tableau**.

Before introducing an example to explain the technique, we will make a few introductory points. Don't worry if you get confused, working through the example will make things clearer.

- a) The simplex method involves testing one feasible solution after another, in a succession of tables or tableaux, until the optimal solution is found. It can be used for problems with any number of decision variables, from two upwards.
- b) In addition to the decision variables, the method introduces additional variables, known as slack variables or surplus variables. There will be one slack (or surplus) variable for each constraint in the problem (excluding non-negativity constraints). For example, if a linear programming problem has three decision variables and four constraints, there will be four slack variables. With the three decision variables, there will therefore be a total of seven variables and four constraints in the problem.
- c) The technique is a **repetitive, step-by-step process**, with each step having the following purposes.
 - (i) To establish a feasible solution (in other words, a feasible combination of decision variable values and slack variable values) and the value of the objective function for that solution.

- (ii) To establish whether that particular solution is one that optimises the value of the objective function.
- d) Each feasible solution is tested by drawing up a **matrix** or **tableau** with the following rows and columns.
- (i) **One row per constraint, plus a solution row**
 - (ii) **One column per decision variable and per slack variable, plus a solution column**
- e) Every variable, whether a decision variable, slack variable or surplus variable, must be ≥ 0 in any feasible solution.
- f) A feature of the simplex method is that if there are n constraints, there will be n variables with a value greater than 0 in any feasible solution. Thus, if there are seven variables in a problem, and four constraints, there will be four variables with a positive value in the solution, and three variables with a value equal to 0.

Keep these points in mind as we work through an example.

Example: the simplex method

An organisation produces and sells two products, X and Y. Relevant information is as follows.

	Materials	Labour	Machine time	Contribution per unit
	<i>units</i>	<i>hours</i>	<i>hours</i>	<i>RWF</i>
X, per unit	5	1	3	20
Y, per unit	2	3	2	16
Total available,				
each week	3,000	1,750	2,100	

Use the simplex method to determine the profit-maximising product mix.

Formulating the problem

We have just two decision variables in this problem, but we can still use the simplex method to solve it.

Step 1 Define variables

Let x be the number of units of X that should be produced and sold.

Let y be the number of units of Y that should be produced and sold.

Step 2 Establish objective function

Maximum contribution (C) = $20x + 16y$ subject to the constraints below.

Step 3 Establish constraints

The constraints are as follows.

Materials	$5x + 2y \leq 3,000$	Machine time	$3x + 2y \leq 2,100$
Labour	$x + 3y \leq 1,750$	Non-negativity	$x \geq 0, y \geq 0$

Step 4 Introduce slack variables

Begin by turning each constraint (ignoring the non-negativity constraints now) into an equation. This is done by introducing slack variables.

Let a be the quantity of unused materials, b be the number of unused labour hours and c be the number of unused machine hours.

Slack variable. ‘Amount of each resource which will be unused if a specific linear programming solution is implemented.’

Question

A problem to be solved using linear programming has three decision variables, six constraints (including two non-negativity constraints) and one objective function.

How many slack variables will be required if the simplex method is used?

- A. 3
- B. 4
- C. 5
- D. 6

Answer

The correct answer is B.

A slack variable is required for each constraint (ignoring non-negativity constraints). There are $6 - 2 = 4$ such constraints.

We can now express the original constraints as equations.

$$5x + 2y + a = 3,000$$

$$x + 3y + b = 1,750$$

$$3x + 2y + c = 2,100$$

The slack variables a, b and c will be equal to 0 in the final solution only if the combined production of X and Y uses up all the available materials, labour hours and machine hours.

Step 5 Values of variables – non-negative or zero?

In this example, there are five variables (x, y, a, b and c) and three equations, and so in any feasible solution that is tested, three variables will have a non-negative value (since there are three equations) which means that two variables will have a value of zero.

Question

A problem to be solved using linear programming has seven variables and four equations based on the original constraints.

How many variables will have a value of zero in any feasible solution determined using the simplex method?

- A. 7
- B. 5
- C. 4
- D. 3

Answer

The correct answer is D.

Four variables will have a non-negative value (since there are four equations), which means that $7 - 4 = 3$ variables will have a value of zero.

Step 6 Express objective function as an equation

It is usual to express the objective function as an equation with the right hand side equal to zero. In order to keep the problem consistent, the slack (or surplus) variables are inserted into the objective function equation, but as the quantities they represent should have no effect on the objective function they are given zero coefficients. In our example, the objective function will be expressed as follows.

Maximise contribution (C) given by $C - 20x - 16y + 0a + 0b + 0c = 0$.

Drawing up the initial tableau and testing the initial feasible solution

We begin by testing a solution that **all the decision variables have a zero value**, and **all the slack variables have a non-negative value**.

Obviously, this is not going to be the optimal solution, but it gives us a starting point from which we can develop other feasible solutions.

Simplex tableaux can be drawn in several different ways, and if you are asked to interpret a given tableau in an examination question, you may need to adapt your understanding of the tableau format in this Study Text to the format in the question. The following points apply to all tableaux, however.

- a) There should be a column for each variable and also a solution column.
- b) It helps to add a further column on the left, to indicate the variable which is in the solution to which the corresponding value in the solution column relates.
- c) There is a row for each equation in the problem, and a solution row.

Here is the initial matrix for our problem. Information on how it has been derived is given below.

<i>Variables in solution</i>	<i>x</i>	<i>y</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>Solution</i>
A	5	2	1	0	0	3,000
B	1	3	0	1	0	1,750
C	3	2	0	0	1	2,100
Solution	-20	-16	0	0	0	0

- a) The figures in each row correspond with the coefficients of the variables in each of the initial constraints. The bottom row or solution row holds the coefficients of the objective function. For example the materials constraint $5x + 2y + a = 3,000$ gives us the first row, 5 (number of x's), 2 (number of y's), 1 (number of a's), then zeros in the b and c columns (since these do not feature in the constraint equation) and finally 3,000 in the solution column.
- b) The variables in the solution are a, b and c (the unused resources).
 - (i) The value of each variable is shown in the solution column. We are testing a solution that all decision variables have a zero value, so there is no production and hence no resources are used. The total resource available is therefore unused.
 - (ii) The column values for each variable in the solution are as follows.
 - 1 in the variable's own solution row
 - 0 in every other row, including the solution row.

- c) The contribution per unit obtainable from x and y is given in the solution row. These are the dual prices or shadow prices of the products X and Y. The minus signs are of no particular significance, except that in the solution given here they have the following meanings.
- (i) A minus shadow price indicates that the value of the objective function can be increased by the amount of the shadow price per unit of the variable that is introduced into the solution, given no change in the current objective function or existing constraints.
 - (ii) A positive shadow price indicates the amount by which the value of the objective function would be decreased per unit of the variable introduced into the solution, given no change in the current objective function or the existing constraints.

Interpreting the tableau and testing for improvement

We can see that the solution is testing $a = 3,000$, $b = 1,750$ and $c = 2,100$, contribution = 0. The co-efficients for the variables not in this solution, x and y, are the dual prices or shadow prices of these variables, given the solution being tested. A negative value to a dual price means that the objective function can be increased; therefore the solution in the tableau is not the optimal solution.

The shadow prices in the initial solution (tableau) indicate the following.

- a) The profit would be increased by RWF20 for every extra unit of x produced (because the shadow price of x is RWF20 per unit).
- b) Similarly, the profit would be increased by RWF16 for every extra unit of y produced (because its shadow price is RWF16 per unit).

Since the solution is not optimal, the contribution may be improved by introducing either x or y into the solution.

The next step

The next step is to test another feasible solution. We do this by introducing one variable into the solution, in the place of one variable that is now removed. In our example, we introduce x or y in place of a, b or c.

The simplex technique continues in this way, producing a feasible solution in each successive tableau, until the optimal solution is reached.

Interpreting the final tableau

If the **shadow prices** on the bottom (solution) row of a tableau are all positive, the tableau shows the optimal solution.

- The solution column shows the optimal production levels and the units of unused resource.
- The figure at the bottom of the solution column/right-hand side of the solution row shows the value of the objective function.
- The figures in the solution row indicate the shadow prices of resources.

After a number of iterations, the following tableau is produced.

<i>Variables in solution</i>	<i>x</i>	<i>y</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>Solution column</i>
X	1	0	0	–	0.4286	400
A	0	0	1	0.2857	–	100
y	0	1	0	0.5714	1.8571	450
Solution row	0	0	0	0.4286	–	15,200
				1.1428	6.2858	

This can be interpreted as follows.

- a) The solution in this tableau is the optimal one, because the shadow prices on the bottom row are all positive.
- b) The optimal solution is to make and sell 400 units of X and 450 units of Y, to earn a contribution of RWF15,200.
- c) The solution will leave 100 units of material unused, but will use up all available labour and machine time.
- d) The shadow price of labour time (b) is RWF1.1428 per hour, which indicates the amount by which contribution could be increased if more labour time could be made available at its normal variable cost.

- e) The shadow price of machine time (c) is RWF6.2858 per hour, which indicates the amount by which contribution could be increased if more machine time could be made available, at its normal variable cost.
- f) The shadow price of materials is nil, because there are 100 units of unused materials in the solution.

Question

TDS manufactures two products, X and Y, which earn a contribution of RWF8 and RWF14 per unit respectively. At current selling prices, there is no limit to sales demand for Y, but maximum demand for X would be 1,200 units. The company aims to maximise its annual profits, and fixed costs are RWF15,000 per annum.

In the year to 30 June 20X2, the company expects to have a limited availability of resources and estimates of availability are as follows.

Skilled labour	maximum 9,000 hours
Machine time	maximum 4,000 hours
Material M	maximum 1,000 tonnes

The usage of these resources per unit of product are as follows.

	X	Y
Skilled labour time	3 hours	4 hours
Machine time	1 hour	2 hours
Material M	½ tonne	¼ tonne

- a) Formulate the problem using the simplex method of linear programming.
- b) Determine how many variables will have a positive value and how many a value of zero in any feasible solution.

Answer

a) The linear programming problem would be formulated as follows.

Define variables

Let x and y be the number of units made and sold of product X and product Y respectively.

Establish objective function

Maximise contribution (C) = $8x + 14y$ subject to the constraints below.

Establish constraints

$$3x + 4y \leq 9,000 \text{ (skilled labour)*}$$

$$x + 2y \leq 4,000 \text{ (machine time)}$$

$$0.5x + 0.25y \leq 1,000 \text{ (material M)}$$

$$x \leq 1,200 \text{ (demand for X)}$$

$$x, y \geq 0$$

* This constraint is that skilled labour hours cannot exceed 9,000 hours, and since a unit of X needs 3 hours and a unit of Y needs 4 hours, $3x + 4y$ cannot exceed 9,000. The other constraints are formulated in a similar way.

Introduce slack variables

Introduce a slack variable into each constraint, to turn the inequality into an equation.

Let a = the number of unused skilled labour hours

b = the number of unused machine hours

c = the number of unused tonnes of material M

d = the amount by which demand for X falls short of 1,200 units

Then

$$3x + 4y + a = 9,000 \text{ (labour hours)}$$

$$x + 2y + b = 4,000 \text{ (machine hours)}$$

$$0.5x + 0.25y + c = 1,000 \text{ (tonnes of M)}$$

$$x + d = 1,200 \text{ (demand for X)}$$

and maximise contribution (C) given by $C - 8x - 14y + 0a + 0b + 0c + 0d = 0$

- b) There are six variables (x, y, a, b, c, d) and four equations. In any feasible solution four variables will have a non-negative value (as there are four equations), while two variables will have a value of zero.

Question

The final tableau to the problem in **Question: formulation of problem** is shown below.

<i>Variables in the solution</i>	<i>x</i>	<i>y</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>Solution column</i>
x	1	0	0	-2	0	0	1,000
y	0	1	-0.5	1.5	0	0	1,500
c	0	0	-0.375	0.625	1	0	125
d	0	0	-1	2	0	1	200
Solution row	0	0	1	5	0	0	29,000

Answer

There is a column in the tableau for every variable, including the slack variables, but the important parts of the tableau are the 'variables in the solution' column, the solution row, and the solution column. These tell us a number of things.

Identifying the variables in the solution

The variables in the solution are x, y, c and d. It follows that a and b have zero values. To be the variable in the solution on a particular row of the table, a value of 1 must appear in the column for that variable, with zero values in every other row of that column. For example, x

is the variable in the solution for the row which has 1 in the x column. There are zeros in every other row in the x column.

The value of the variables

The solution **column** gives the value of each variable.

x	1,000	(units made of X)
y	1,500	(units made of Y)
c	125	(unused material M)
d	200	(amount below the 1,200 maximum of demand for X)

This means that contribution will be maximised by making and selling 1,000 units of X and 1,500 units of Y. This will leave 125 unused tonnes of material M, and production and sales of X will be 200 units below the limit of sales demand. Since a and b are both zero, there is no unused labour and machine time; in other words, all the available labour and machine hours will be fully utilised.

The total contribution

The value of the objective function – here, the total contribution – is in both the solution row and the solution column. Here it is RWF29,000.

Shadow prices

The solution row gives the **shadow prices** of each variable. Here, the shadow price of a is RWF1 per labour hour and that for b is RWF5 per machine hour.

This means that if more labour hours could be made available at their normal variable cost per hour, total contribution could be increased by RWF1 per extra labour hour. Similarly, if more machine time could be made available, at its normal variable cost, total. Here is the final tableau of a problem involving the production of products X and Y solved using the simplex method of linear programming.

<i>Variables in solution</i>	<i>x</i>	<i>y</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>Solution column</i>
x	1	0	-2.0	0	3.0	0	0	550
y	0	1	-0.8	0	0.5	0	0	720
b	0	0	1.5	1	1.0	0	0	95
d	0	0	0.7	0	-1.1	1	0	50
e	0	0	2.0	0	1.8	0	1	104
Solution row	0	0	7.0	0	4.0	0	0	14,110

Draw a ring around the column or row which shows the variables in the solution.

Answer

<i>Variables in solution</i>	<i>x</i>	<i>y</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>Solution column</i>
x	1	0	-2.0	0	3.0	0	0	550
y	0	1	-0.8	0	0.5	0	0	720
b	0	0	1.5	1	1.0	0	0	95
d	0	0	0.7	0	-1.1	1	0	50
e	0	0	2.0	0	1.8	0	1	104
Solution row	0	0	7.0	0	4.0	0	0	14,110

To be a variable in the solution, a value of 1 must appear in the column for the variable, with zero values in every other row. Refer to the tableau in **Question: identification of variables**.

What is the profit-maximising product mix?

- A. Make 95 units of B, 50 units of D and 104 units of E
- B. Make 550 units of X and 720 units of Y
- C. Make 4 units of C and 7 units of A
- D. None of the above

The correct answer is B. The answer can be found in the solution column in the rows for x and y.

Question

Refer to the tableau in Question: identification of variables. Suppose that variables a to e refer to the unused quantity of resources A to E.

Fill in the blank in the sentence below.

..... units of resource A will be unused.

Answer

The correct answer is that 0 units of A will be unused. A has a zero value in the solution column and so resource A is fully used.

Question

Refer to the tableau in Question: identification of variables. The shadow price of resource C is RWF3. True or false?

Answer

The correct answer is RWF4, so the statement is false.

The solution row gives the shadow price of each variable.

E. SENSITIVITY ANALYSIS

You might be asked to carry out some sensitivity analysis on a simplex tableau giving the optimal solution to a linear programming problem. This could involve the following.

- a) Testing how the optimal solution would change if there were either more or less of a scarce resource.
- b) Testing whether it would be worthwhile obtaining more of a scarce resource by paying a premium for the additional resources, for example by paying an overtime premium for extra labour hours, or by paying a supplier a higher price for extra raw materials.

The effect of having more or less of a scarce resource

Sensitivity analysis can be applied to the final tableau to determine the effect of having more or less of a scarce resource (indicated by figures in the column for the resource's slack variable).

The optimal solution to a linear programming problem is based on the assumption that the constraints are known with certainty, and fixed in quantity. Sensitivity analysis enables us to test how the solution would alter if the quantity of a scarce resource (the size of a constraint) were to change.

Example: the effect of having more or less of a scarce resource

Return to our previous example in which both labour hours and machine hours are fully used. How would the solution change if more labour hours (variable b) were available?

Solution

The simplex tableau, and in particular the figures in the b column, provide the following information for each extra labour hour that is available.

- a) The contribution would increase by RWF1.1428
- b) The value of x would fall by 0.2857 units
- c) The value of a (unused materials) would increase by 0.5714 units
- d) The value of y would increase by 0.4286 units

In other words, we would be able to make 0.4286 units of Y extra, to earn contribution of (x RWF16) RWF6.8576, but we would make 0.2857 units less of X and so lose contribution of (x RWF20) RWF5.714, leaving a net increase in contribution of RWF(6.8576 – 5.714) = RWF1.1436. Allowing for rounding errors of RWF0.0008, this is the figure already given above for the increase in contribution.

Since $x = 400$ in the optimal tableau, and extra labour hours would lead to a reduction of 0.2857 units of x, there is a limit to the number of extra labour hours that would earn an extra RWF1.1428. This limit is calculated as $400/0.2857 = 1,400$ extra labour hours.

In other words, the shadow price of RWF1.1428 per hour for labour is only valid for about 1,400 extra labour hours on top of the given constraint in the initial problem, which was 1,750 hours, (that is up to a total limit of 3,150 hours).

If there were fewer labour hours available, the same sort of analysis would apply, but in reverse.

- a) The contribution would fall by RWF1.1428 per hour unavailable
- b) The value of x would increase by 0.2857 units
- c) The value of a would fall by 0.5714 units
- d) The value of y would fall by 0.4286 units

Example: obtaining extra resources at a premium on cost

Sensitivity analysis can also be applied to test whether or not it would be **worthwhile to obtain more of a scarce resource** by paying a premium for additional supplies (only if the shadow price is greater than the additional cost).

Suppose we are given the following additional information about our example.

- a) The normal variable cost of labour hours (variable b) is RWF4 per hour, but extra labour hours could be worked in overtime, when the rate of pay would be time-and-a-half.
- b) The normal variable cost of machine time is RWF1.50 per hour, but some extra machine time could be made available by renting another machine for 40 hours per week, at a rental cost of RWF160. Variable running costs of this machine would be RWF1.50 per hour.

Would it be worth obtaining the extra resources?

Solution

We know that the shadow price of labour hours is RWF1.1428 and of machine hours is RWF6.2858. We can therefore deduce the following.

- a) Paying an overtime premium of RWF2 per hour for labour would not be worthwhile, because the extra contribution of RWF1.1428 per hour would be more than offset by the cost of the premium, leaving the company worse off by RWF0.8572 per hour worked in overtime.
- b) Renting the extra machine would be worthwhile, but only by RWF91.43 (which is perhaps too small an amount to bother with).

	RWF
Extra contribution from 40 hours of machine time (x RWF6.2858)	251.43
Rental cost	<u>160.00</u>
Net increase in profit	<u>91.43</u>

Note that the variable running costs do not enter into this calculation since they are identical to the normal variable costs of machine time. We are **concerned here only with the additional costs.**

Question

An organisation manufactures three products, tanks, trays and tubs, each of which passes through three processes, X, Y and Z.

<i>Process</i>	Process hours per unit			<i>Total process</i>
	<i>Tanks</i>	<i>Trays</i>	<i>Tubs</i>	<i>hours available</i>
X	5	2	4	12,000
Y	4	5	6	24,000
Z	3	5	4	18,000

The contribution to profit of each product are RWF2 for each tank, RWF3 per tray and RWF4 per tub.

Required

Fill in the blanks in (a) and (b) below, which relate to the formulation of the above data into a simplex linear programming model. Use the following notation.

Let a be the number of units of tanks produced

b be the number of units of trays produced

c be the number of units of tubs produced

x = quantity of unused process X hours

y = quantity of unused process Y hours

z = quantity of unused process Z hours

- a) Maximise contribution (C) given by subject to the following constraints in (b).
- b) (process X hours)
..... (process Y hours)
..... (process Z hours)

Answer

- a) C is given by $C - 2a - 3b - 4c + 0x + 0y + 0z$
- b) Constraint for process X hours: $5a + 2b + 4c + x = 12,000$
Constraint for process Y hours: $4a + 5b + 6c + y = 24,000$
Constraint for process Z hours: $3a + 5b + 4c + z = 18,000$

Question

The final simplex tableau, based on the data in the question above, looks like this.

<i>Variables in solution</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>x</i>	<i>y</i>	<i>z</i>	<i>Solution column</i>
c	1.583	0	1	0.417	0	-0.167	2,000
y	-2.167	0	0	-0.833	1	-0.667	2,000
b	-0.667	1	0	-0.333	0	0.333	2,000
Solution row	2.333	0	0	0.667	0	0.333	14,000

Required

- Determine how many of each product should be produced and the maximum contribution. Calculate how much slack time, if any, is available in the processes.
- Explain how your solution would vary if an extra 3,000 hours of process X time could be made available.
- Describe what would happen to the production schedule and budgeted contribution if an order were received for 300 units of tanks which the company felt that it had to accept, because of the importance of the customer. **Ignore** the increase of process X time in part (b) above.

Answer

- Contribution is maximised at RWF14,000 by making 2,000 units of tubs and 2,000 units of trays. No tanks would be made.

There will be 2,000 slack hours in process Y. Process X and process Z hours will be fully utilised.

- The shadow price of process X time is RWF0.667 per hour, and for every extra hour of process X time that can be made available (at its normal variable cost), the production quantities could be altered in such a way that the following would happen.
 - Contribution would go up by RWF0.667 per extra process X hour used.
 - c (the quantity of tubs) would go up by 0.417 units.
 - b (the quantity of trays) would go down by 0.333 units.
 - y (unused process Y time) would fall by 0.833 hours.

This is **only true up to the point** where so many extra process X hours have been made available that either b or y reaches 0 in value. This will be at the following points.

(i) For y, after $\frac{2,000}{0.833} = 2,400$ extra process X hours

(ii) For b, after $\frac{2,000}{0.333} = 6,000$ extra process X hours

2,400 is the lowest of these two limits.

The shadow price is therefore valid only for up to 2,400 extra process X hours, so that the full 3,000 available would not be required.

The **new optimal solution** would therefore be to make and sell the following.

c $2,000 + (2,400 \times 0.417) = 3,000$ units

b $2,000 - (2,400 \times 0.333) = 1,200$ units

These would require a total of 14,400 hours in process X, 24,000 hours in process Y and 18,000 hours in process Z.

Contribution would be as follows.

	RWF
Tubs 3,000 x RWF4	12,000
Trays 1,200 x RWF3	<u>3,600</u>
	15,600
Contribution in initial solution	<u>14,000</u>
Increase in contribution (2,400 x RWF0.667)	<u>1,600</u>

c) Going back to the original solution, if an order is received for 300 units of tanks, the production schedule would be re-arranged so that **for each unit of tank made the following would happen.**

(i) Contribution would fall by RWF2.333.

(ii) 1.583 units less of tubs (variable c) would be made.

- (iii) 0.667 units more of trays (variable b) would be made.
- (iv) Unused process Y time would increase by 2.167 hours.

The new production and contribution budget would be as follows.

Product	Units	<i>Process X</i>	<i>Process Y</i>	<i>Process Z</i>	<i>Contribution</i>
		<i>time</i>	<i>time</i>	<i>time</i>	
		Hours	Hours	Hours	RWF
Tanks (a)	300	1,500	1,200	900	600
Trays (b)	2,200*	4,400	11,000	11,000	6,600
Tubs (c)	1,525**	<u>6,100</u>	<u>9,150</u>	<u>6,100</u>	<u>6,100</u>
		<u>12,000</u>	<u>21,350</u>	<u>18,000</u>	<u>13,300</u>

The contribution is RWF700 lower than in the original optimal solution (which represents 300 tanks x RWF2.333).

Unused process Y time is 2,650 hours, which is 650 more than in the original solution (which represents 300 x 2.167)

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F. USING COMPUTER PACKAGES

Spreadsheet packages can be used to solve linear programming problems.

- The **slack/surplus** columns provide information about the slack values of \leq constraints and the surplus values of any \geq constraints.
- The **worth** column shows the positive shadow price of resources.
- The **relative loss** shows by how much contribution (usually) would fall if extra units of particular decision variables were produced.

Nowadays, modern spreadsheet packages can be used to solve linear programming problems.

Suppose an organisation produces three products, X and Y and Z, subject to four constraints (1, 2, 3, 4).

- a) Constraints 1 and 2 are 'less than or equal to' resource constraints.
- b) Constraint 3 provides a limit on the number of X that can be produced.
- c) Constraint 4 is a 'greater than or equal to' constraint and provides for a minimum number of Z to be produced (400).

The organisation wishes to maximise contribution.

Typical output from a spreadsheet package for such a problem is shown below.

<i>Objective function (c)</i>		137,500
<i>Variable</i>	<i>Value</i>	<i>Relative loss</i>
x	475.000	0.000
y	0.000	105.000
z	610.000	0.000
<i>Constraint</i>	<i>Slack/surplus</i>	<i>Worth</i>
1	17.000	0.000
2	0.000	290.000
3	0.000	1,150.000
4	210.000	0.000

Interpretation

- a) Total optimal contribution (c) will be RWF137,500.
- b) The variable and value columns mean that $x = 475$, $y = 0$ and $z = 610$.

To maximise contribution, 475 units of X and 610 units of Z should therefore be produced. No units of Y should be produced.

- c) The constraint and slack/surplus columns provide information about the slack values of 'less than or equal to' constraints and the surplus values for any 'greater than or equal to' constraints.
 - (i) Constraint 1 is a 'less than or equal to' resource constraint. The slack is 17 and so 17 units of resource 1 will be unused in the optimal solution.
 - (ii) Constraint 2 is a 'less than or equal to' resource constraint. The slack is zero, indicating that all available resource 2 will be used in the optimal solution.
 - (iii) Constraint 3 provides a limit on x. The slack is zero, showing that the limit has been met.
 - (iv) Constraint 4 provides for a minimum z. The surplus is 210, meaning $400 + 210 = 610$ units of Z are made.
- d) Worth. This column shows the positive shadow price of resources (the amount that contribution (or, in general terms, c) alters if the availability of the resource is changed by one unit).
 - (i) Contribution would increase by RWF290 if one extra unit of resource 2 were made available.
 - (ii) Contribution would increase by RWF1,150 if the limit on the minimum number of Z to be produced altered by 1.
 - (iii) Resource 1 has a worth of 0 because 17 units of the resource are unused in the optimal solution.

In general, any constraint with a slack of zero has a positive worth figure, while any constraint with a positive slack figure will have a worth of zero.

- e) Relative loss. This indicates that if one unit of Y were produced, total contribution (or generally c) would fall by RWF105. A relative loss of RWF105 would therefore be made for every unit of Y made. Units of Y should only be made if unit contribution of Y increases by RWF105.

X and Z have relative losses of zero, indicating that they should be made.

In general, only those decision variables with a relative loss of zero will have a positive value in the optimal solution.

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G. USING LINEAR PROGRAMMING

There are a number of assumptions and practical difficulties in the use of linear programming.

Further assumptions

In addition, there are further assumptions if we are dealing with product mix decisions involving several limiting factors.

- a) The **total amount available of each scarce resource is known with accuracy.**
- b) There is **no interdependence between the demand** for the different products or services, so that there is a completely free choice in the product or service mix without having to consider the consequences for demand or selling prices per unit.

In spite of these assumptions, linear programming is a useful technique in practice. Some statistical studies have been carried out suggesting that linear cost functions do apply over fairly wide ranges of output, and so the assumptions underlying linear programming may be valid.

Uses of linear programming

- a) **Budgeting.** If scarce resources are ignored when a budget is prepared, the budget is unattainable and is of little use for planning and control. When there is more than one scarce resource, linear programming can be used to identify the most profitable use of resources.
- b) **Calculation of relevant costs.** The calculation of relevant costs is essential for decision making. The **relevant cost** of a scarce resource is calculated as **acquisition cost of the resource plus opportunity cost**. When more than one scarce resource exists, the opportunity cost (or shadow price) should be established using linear programming techniques.
- c) **Selling different products.** Suppose that an organisation faced with resource constraints manufactures products X and Y and linear programming has been used to determine the shadow prices of the scarce resources. If the organisation now wishes to manufacture and sell a modified version of product X (Z), requiring inputs of the scarce resources, the relevant costs of these scarce resources can be determined (see

above) to ascertain whether the production of X and Y should be restricted in order to produce Z

- d) **Maximum payment for additional scarce resources.** This use of shadow prices has been covered in this chapter.
- e) **Control.** Opportunity costs are also important for cost control: standard costing can be improved by incorporating opportunity costs into variance calculations. For example, adverse material usage variances can be an indication of material wastage. Such variances should be valued at the standard cost of the material plus the opportunity cost of the loss of one scarce unit of material. Such an approach highlights the true cost of the inefficient use of scarce resources and encourages managers of responsibility centres to pay special attention to the control of scarce factors of production. For organisations using an optimised production technology (OPT) strategy, this approach is particularly useful because variances arising from bottleneck operations will be reported in terms of opportunity cost rather than purchase cost.
- f) **Capital budgeting.** Linear programming can be used to determine the combination of investment proposals that should be selected if investment funds are restricted in more than one period.

Practical difficulties with using linear programming

Difficulties with applying the linear programming technique in practice include the following.

- a) It may be difficult to identify which resources are likely to be in short supply and what the amount of their availability will be.

With linear programming, the profit-maximising product mix and the shadow price of each limiting factor depend on the total estimated availability of each scarce resource. So it is not sufficient to know that labour hours and machine hours will be in short supply, it is also necessary to guess how many labour hours and machine hours will be available. Estimates of future availability will inevitably be prone to inaccuracy and any such inaccuracies will invalidate the profit-maximising product mix derived from the use of linear programming.

- b) Management may **not make product mix decisions which are profit-maximising**. They may be more concerned to develop a production/sales plan which has the following features.
- (i) Realistic
 - (ii) Acceptable to the individual managers throughout the organisation
 - (iii) Acceptable to the rest of the workforce
 - (iv) Promises a 'satisfactory' profit and accounting return

In other words, management might look for a **satisfactory product mix** which achieves a satisfactory return, sales revenue and market share whilst at the same time plans operations and targets of achievement which employees can accept as realistic, not too demanding and unreasonable, and not too threatening to their job security.

If a 'satisfactory' output decision is adopted, the product mix or service mix **recommended by the linear programming** (profit-maximising) technique will inevitably be 'watered down', amended or ignored.

- c) **The assumption of linearity may be totally invalid except over smaller ranges.** For example, in a profit maximisation problem, it may well be found that there are substantial changes in unit variable costs arising from increasing or decreasing returns to scale.
- d) The linear programming model is essentially **static** and is therefore not really suitable for analysing in detail the effects of changes in the various parameters, for example over time.
- e) In some circumstances, a practical solution derived from a linear programming model may be of **limited use** as, for example, where the variables may only take on **integer values**. A solution must then be found by a combination of rounding up and trial and error.
- f) The **shadow price** of a scarce resource **only applies up to a certain limit**.

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STUDY UNIT 17

Risk and Uncertainty

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A. RISK AND UNCERTAINTY

An example of a risky situation is one in which we can say that there is a 70% probability that returns from a project will be in excess of RWF100,000 but a 30% probability that returns will be less than RWF100,000. If we cannot predict an outcome or assign probabilities, we are faced with an uncertain situation.

Risk involves situations or events which may or may not occur, but whose probability of occurrence can be calculated statistically and the frequency of their occurrence predicted from past records. Thus insurance deals with risk.

Uncertain events are those whose outcome cannot be predicted with statistical confidence.

In everyday usage the terms risk and uncertainty are not clearly distinguished. If you are asked for a definition, do not make the mistake of believing that the latter is a more extreme version of the former. It is not a question of degree, it is a question of whether or not sufficient information is available to allow the lack of certainty to be quantified. As a rule, however, the terms are used interchangeably.

Risk preference

People may be **risk seekers**, **risk neutral** or **risk averse**.

A **risk seeker** is a decision maker who is interested in the best outcomes no matter how small the chance that they may occur.

A decision maker is **risk neutral** if he is concerned with what will be the most likely outcome.

A **risk averse** decision maker acts on the assumption that the worst outcome might occur.

This has clear implications for managers and organisations. A risk seeking manager working for an organisation that is characteristically risk averse is likely to make decisions that are not congruent with the goals of the organisation. There may be a role for the management accountant here, who could be instructed to present decision-making information in such a way as to ensure that the manager considers all the possibilities, including the worst.

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B. ALLOWING FOR UNCERTAINTY

Management accounting directs its attention towards the **future** and the future is **uncertain**. For this reason a number of methods of taking uncertainty into consideration have evolved.

Research techniques to reduce uncertainty

Market research can be used to reduce uncertainty.

Market research is the systematic process of gathering, analysing and reporting data about markets to investigate, describe, measure, understand or explain a situation or problem facing a company or organisation.

Market research **involves tackling problems**. The assumption is that these problems can be solved, no matter how complex the issues are, if the researcher follows a line of enquiry in a systematic way, without losing sight of the main objectives. Gathering and analysing all the facts will ultimately lead to **better decision making**.

The role of market research

In the last 20 years or so market research has become a much more widespread activity. Organisations – in the private sector, the public sector and the not-for-profit sector – rely on research to inform and improve their **planning and decision making**.

Market research enables organisations to understand the needs and opinions of their customers and other stakeholders. Armed with this knowledge they are able to make better quality decisions and provide better products and better services.

Thus, research influences what is provided and the way it is provided. It **reduces uncertainty and monitors performance**. A management team which possesses accurate information relating to the marketplace will be in a strong position to make the best decisions in an increasingly competitive world.

Decision-makers need data to reduce **uncertainty** and **risk** when planning for the future and to monitor business performance. Market researchers provide the data that helps them to do this.

Types of data collected

Data can be either **primary** (collected at first hand from a sample of respondents), or **secondary** (collected from previous surveys, other published facts and opinions, or from experts). Secondary research is also known as **desk research**, because it can be carried out from one's desk.

More importantly for research practice and analysis, data can be either quantitative or qualitative.

Quantitative data usually deals with numbers and typically provides the decision maker with information about **how many** customers, competitors etc act in a certain way. Quantitative data can, for example, tell the researcher **what** people need or consume, or **where, when** and **how** people buy goods or consumer services.

Qualitative data tells us **why** consumers think/buy or act the way they do. Qualitative data is used in **consumer insight** (eg understanding what makes consumers prefer one brand to another), **media awareness** (eg how much of an advertisement is noticed by the public), **new product development** studies and for many other reasons.

Qualitative research has as its specific purpose the uncovering and understanding of thought and opinion. It is carried out on relatively small samples and unstructured or semi-structured techniques, such as individual in depth interviews and group discussions (also known as **focus groups**), are used.

Conservatism

This approach simply involves estimating outcomes in a conservative manner in order to provide a built-in safety factor.

However, the method fails to consider explicitly a **range** of outcomes and, by concentrating only on conservative figures, may also fail to consider the **expected** or most likely outcomes.

Conservatism is associated with **risk aversion** and prudence (in the general sense of the word). In spite of its shortcomings it is probably the **most widely used** method in practice.

Worst/most likely/best outcome estimates

A more scientific version of conservatism is to measure the most likely outcome from a decision, and the worst and best possible outcomes. This will show the **full range of possible outcomes** from a decision, and might help managers to reject certain alternatives because the worst possible outcome might involve an unacceptable amount of loss. This requires the preparation of **pay-off tables**.

Pay-off tables

Pay-off tables **identify and record all possible outcomes (or pay-offs)** in situations where the action taken affects the outcomes.

Example: worst/best possible outcomes

Omelette Co is trying to set the sales price for one of its products. Three prices are under consideration, and expected sales volumes and costs are as follows.

<i>Price per unit</i>	<i>RWF4</i>	<i>RWF4.30</i>	<i>RWF4.40</i>
Expected sales volume (units)			
Best possible	16,000	14,000	12,500
Most likely	14,000	12,500	12,000
Worst possible	10,000	8,000	6,000

Which price should be chosen?

Solution

Here we need to prepare a pay-off table showing pay-offs (contribution) dependant on different levels of demand and different selling prices.

<i>Price per unit</i>	<i>RWF4</i>	<i>RWF4.30</i>	<i>RWF4.40</i>
Contribution per unit	RWF2	RWF2.30	RWF2.40
<i>Total contribution towards fixed costs</i>	RWF	RWF	RWF
Best possible	32,000	32,200	30,000
Most likely	28,000	28,750	28,800
Worst possible	20,000	18,400	14,400

- a) The highest contribution based on most likely sales volume would be at a price of RWF4.40 but arguably a price of RWF4.30 would be much better than RWF4.40, since the most likely profit is almost as good, the worst possible profit is not as bad, and the best possible profit is better.
- b) However, only a price of RWF4 guarantees that the company would not make a loss, even if the worst possible outcome occurs. (Fixed costs of RWF20,000 would just be covered.) A risk averse management might therefore prefer a price of RWF4 to either of the other two prices.

C. PROBABILITIES AND EXPECTED VALUES

Expected values indicate what an outcome is likely to be in the long term with repetition. Fortunately, many business transactions do occur over and over again.

Although the outcome of a decision may not be certain, there is some likelihood that probabilities could be assigned to the various possible outcomes from an analysis of previous experience.

Expected values

Where probabilities are assigned to different outcomes we can evaluate the worth of a decision as the **expected value**, or weighted average, of these outcomes. The principle is that when there are a number of alternative decisions, each with a range of possible outcomes, the optimum decision will be the one which gives the highest expected value.

Example: expected values

Suppose a manager has to choose between mutually exclusive options A and B, and the probable outcomes of each option are as follows.

<i>Option A</i>		<i>Option B</i>	
<i>Probability</i>	<i>Profit</i>	<i>Probability</i>	<i>Profit</i>
	RWF		RWF
0.8	5,000	0.1	(2,000)
0.2	6,000	0.2	5,000
		0.6	7,000
		0.1	8,000

The expected value (EV) of profit of each option would be measured as follows.

Option A			Option B		
Prob	Profit	EV of profit	Prob	Profit	EV of profit
	<i>RWF</i>	<i>RWF</i>		<i>RWF</i>	<i>RWF</i>
0.8	x 5,000	= 4,000	0.1	x (2,000)	= (200)
0.2	x 6,000	= 1,200	0.2	x 5,000	= 1,000
	EV	= 5,200	0.6	x 7,000	= 4,200
			0.1	x 8,000	= 800
				EV	= 5,800

In this example, since it offers a higher EV of profit, option B would be selected in preference to A, unless further risk analysis is carried out.

Question

A manager has to choose between mutually exclusive options C and D and the probable outcomes of each option are as follows.

<i>Option C</i>		<i>Option D</i>	
<i>Probability</i>	<i>Cost</i>	<i>Probability</i>	<i>Cost</i>
	<i>RWF</i>		<i>RWF</i>
0.29	15,000	0.03	14,000
0.54	20,000	0.30	17,000
0.17	30,000	0.35	21,000
		0.32	24,000

Both options will produce an income of RWF30,000. Which should be chosen?

Answer

Option C. Do the workings yourself in the way illustrated above. Note that the probabilities are for costs not profits.

Limitations of expected values

The preference for B over A on the basis of expected value is marred by the fact that A's **worst possible** outcome is a profit of RWF5,000, whereas B might incur a loss of RWF2,000 (although there is a 70% chance that profits would be RWF7,000 or more, which would be more than the best profits from option A).

Since the decision must be made **once only** between A and B, the expected value of profit (which is **merely a weighted average** of all possible outcomes) has severe limitations as a decision rule by which to judge preference. The expected value will **never actually occur**.

Expected values are used to support a **risk-neutral attitude**. A risk-neutral decision maker will ignore any variability in the range of possible outcomes and be concerned only with the expected value of outcomes.

Expected values are more valuable as a guide to decision making where they refer to outcomes which will occur **many times over**. Examples would include the probability that so many customers per day will buy a can of baked beans, the probability that a customer services assistant will receive so many phone calls per hour, and so on.

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D. DECISION RULES

The 'play it safe' basis for decision making is referred to as the **maximin basis**. This is short for '**maximise the minimum achievable profit**'.

A basis for making decisions by looking for the best outcome is known as the **maximax basis**, short for '**maximise the maximum achievable profit**'.

The 'opportunity loss' basis for decision making is known as **minimax regret**.

The maximin decision rule

The maximin decision rule suggests that a decision maker should select the alternative that offers the least unattractive worst outcome. This would mean choosing the alternative that maximises the minimum profits.

Suppose a businessman is trying to decide which of three mutually exclusive projects to undertake. Each of the projects could lead to varying net profit under three possible scenarios.

		<i>Profits</i>		
		<i>Project</i>		
		<i>D</i>	<i>E</i>	<i>F</i>
Scenarios	I	100	80	60
	II	90	120	85
	III	(20)	10	85

The maximin decision rule suggests that he should select the 'smallest worst result' that could happen. This is the decision criterion that managers should 'play safe' and either minimise their losses or costs, or else go for the decision which gives the higher minimum profits. If he selects project D the worst result is a loss of 20. The worst results for E and F are profits of 10 and 60 respectively. The best worst outcome is 60 and project F would therefore be selected (because this is a better 'worst possible' than either D or E).

Criticisms of maximin

- It is defensive and conservative, being a safety first principle of avoiding the worst outcomes without taking into account opportunities for maximising profits.
- It ignores the probability of each different outcome taking place.

Maximax

The **maximax criterion** looks at the best possible results. Maximax means 'maximise the maximum profit'.

Using the information in Section 4.1 above, the maximum profit for D is 100, for E is 120 and for F is 85.

Project E would be chosen if the maximax rule is followed.

Criticisms of maximax

- a) It ignores probabilities.
- b) It is **over-optimistic**.

Question

A company is considering which one of three alternative courses of action, A, B and C to take. The profit or loss from each choice depends on which one of four economic circumstances, I, II, III or IV will apply. The possible profits and losses, in thousands of Rwandan francs, are given in the following payoff table. Losses are shown as negative figures.

		<i>Action</i>		
		<i>A</i>	<i>B</i>	<i>C</i>
Circumstance	I	70	60	70
	II	-10	20	-5
	III	80	0	50
	IV	60	100	115

Required

State which action would be selected using each of the maximax and maximin criteria.

Answer

- a) The **best possible outcomes** are as follows.

A (circumstance III): 80

B (circumstance IV): 100

C (circumstance IV): 115

As 115 is the highest of these three figures, action C would be chosen using the maximax criterion.

b) The **worst possible outcomes** are as follows.

A (circumstance II): -10

B (circumstance III): 0

C (circumstance II): -5

The best of these figures is 0 (neither a profit nor a loss), so action B would be chosen using the maximin criterion.

Minimax regret rule

The **minimax regret rule** aims to minimise the regret from making the wrong decision. Regret is the opportunity lost through making the wrong decision.

We first consider the extreme to which we might come to regret an action we had chosen.

Regret for any combination of action and circumstances = Profit for best action in those circumstances – Profit for the action actually chosen in those circumstances

The minimax regret decision rule is that the decision option selected should be the one which minimises the maximum potential regret for any of the possible outcomes.

Using the example in Section 4.1, a table of regrets can be compiled as follows.

		<i>Project</i>		
		<i>D</i>	<i>E</i>	<i>F</i>
<i>Scenario</i>	I	0	20*	40**
	II	30***	0	35
	III	<u>105</u>	<u>75</u>	<u>0</u>
Maximum regret		<u>105</u>	<u>75</u>	<u>40</u>

The lowest of maximum regrets is 40 with project F so project F would be selected if the minimax regret rule is used.

Contribution tables

Questions requiring application of the decision rules often incorporate a number of variables, each with a range of possible values. For example these variables might be:

- Unit price and associated level of demand
- Unit variable cost

Each variable might have, for example, three possible values.

Before being asked to use the decision rules, exam questions could ask you to **work out contribution** for each of the possible outcomes. (Alternatively profit figures could be required if you are given information about fixed costs.)

The number of possible outcomes = number of values of variable 1 x number of values of variable 2 x number of values of variable 3 etc

So, for example, if there are **two** variables, each with **three** possible values, there are **3 x 3 = 9 outcomes**.

Perhaps the easiest way to see how to draw up contribution tables is to look at an example.

Example: contribution tables and the decision rules

Suppose the budgeted demand for product X will be 11,500 units if the price RWF10, 8,500 units if the price is RWF12 and 5,000 units if the price is RWF14. Variable costs are estimated at either RWF4, RWF5, or RWF6 per unit. A decision needs to be made on the price to be charged.

Here is a contribution table showing the budgeted contribution for each of the nine possible outcomes.

<i>Demand</i>	<i>Price</i>	<i>Variable cost</i>	<i>Unit contribution</i>	<i>Total contribution</i>
	RWF	RWF	RWF	RWF'000
11,500	10	4	6	69.0
11,500	10	5	5	57.5
11,500	10	6	4	46.0
8,500	12	4	8	68.0
8,500	12	5	7	59.5
8,500	12	6	6	51.0
5,000	14	4	10	50.0
5,000	14	5	9	45.0
5,000	14	6	8	40.0

Once the table has been drawn up, the decision rules can be applied.

Solution

Maximin

We need to maximise the minimum contribution.

Demand/price	Minimum contribution
11,500/RWF10	RWF46,000
8,500/RWF12	RWF51,000
5,000/RWF14	RWF40,000

Set a price of RWF12.

Maximax

We need to maximise the maximum contribution.

Demand/price	Maximum contribution
11,500/RWF10	RWF69,000
8,000/RWF12	RWF68,000
5,000/RWF14	RWF50,000

Set a price of RWF10.

Minimax regret

We need to minimise the maximum regret (lost contribution) of making the wrong decision.

Variable cost	Price		
	RWF10	RWF12	RWF14
RWF			
4	–	RWF1,000	RWF19,000
5	RWF2,000	–	RWF14,500
6	RWF5,000	–	RWF11,000
Minimax regret	RWF5,000	RWF1,000	RWF19,000

Minimax regret strategy (**price of RWF12**) is that which minimises the maximum regret (RWF1,000).

Sample working

At a variable cost of RWF4, the best strategy would be a price of RWF10. Choosing a price of RWF12 would mean lost contribution of RWF69,000 – RWF68,000, while choosing a price of RWF14 would mean lost contribution of RWF69,000 – RWF50,000.

E. DECISION TREES

Decision trees are diagrams which illustrate the choices and possible outcomes of a decision.

Rollback analysis evaluates the EV of each decision option. You have to work from right to left and calculate Evs at each outcome point.

A probability problem such as ‘what is the probability of throwing a six with one throw of a dice? Is fairly straightforward and can be solved using the basic principles of probability.

More complex probability questions, although solvable using the basic principles, require a clear logical approach to ensure that all possible choices and outcomes of a decision are taken into consideration.

Decision trees are a useful means of interpreting such probability problems.

A **decision tree** is a pictorial method of showing a sequence of interrelated decisions and their expected outcomes. Decision trees can incorporate both the probabilities of, and values of, expected outcomes, and are used in decision-making

Exactly how does the use of a decision tree permit a clear and logical approach?

- All the possible choices that can be made are shown as branches on the tree.
- All the possible outcomes of each choice are shown as subsidiary branches on the tree.

Constructing a decision tree.

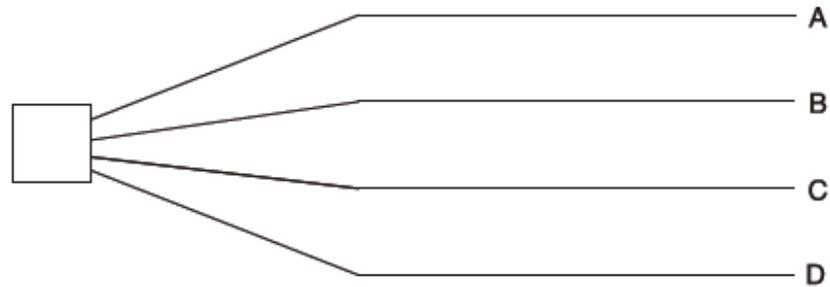
There are two stages in preparing a decision tree.

- Drawing the tree itself to show all the choices and outcomes
- Putting in the numbers (the probabilities, outcome values and EVs)

Every decision tree starts from a **decision point** with the **decision options** that are currently being considered.

- a) It helps to identify the **decision point**, and any subsequent decision points in the tree, with a symbol. Here, we shall use a **square shape**.
- b) There should be a **line**, or **branch**, for each **option** or **alternative**

It is conventional to draw decision trees from left to right, and so a decision tree will start as follows.



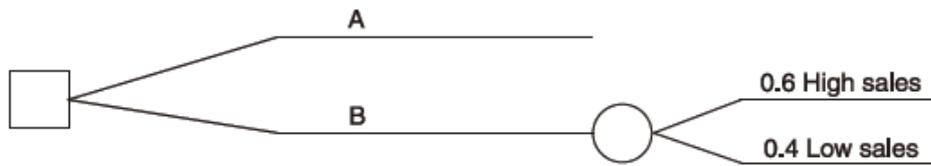
The **square** is the **decision point**, and A, B, C, and D represent **four alternatives** from which a choice must be made (such as buy a new machine with cash, hire a machine, continue to use existing machine, raise a loan to buy a machine).

If the outcome from any choice is certain, the branch of the decision tree for that alternative is complete.

If the outcome of a particular choice is uncertain, the various possible outcomes must be shown.

We show the various possible outcomes on a decision tree by inserting an **outcome point** on the **branch** of the tree. Each possible outcome is then shown as a **subsidiary branch**, coming out from the outcome point. The probability of each outcome occurring should be written on the branch of the tree which represents that outcome.

To distinguish decision points from outcome points, a **circle will be used as the symbol for an outcome point**.



In the example above, there are two choices facing the decision-maker, A and B. The outcome if A is chosen is known with certainty, but if B is chosen, there are two possible outcomes, high sales (0.6 probability) or low sales (0.4 probability).

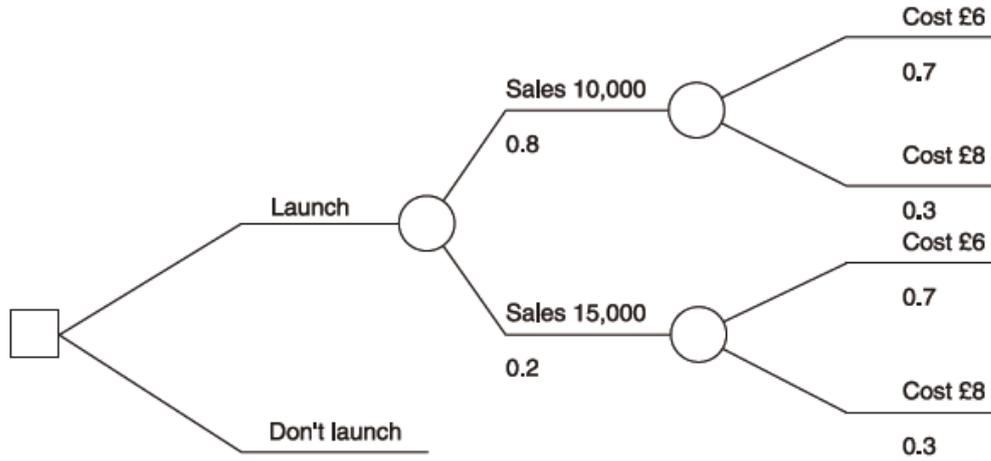
When several outcomes are possible, it is usually simpler to show two or more stage of outcome points on the decision tree.

Example: Several possible outcomes

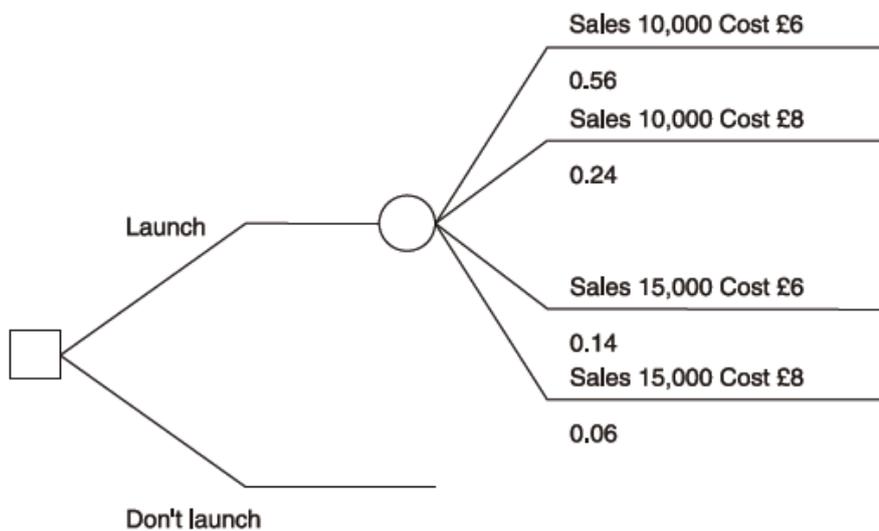
A company can choose to launch a new product XYZ or not. If the product is launched, expected sales and expected unit costs might be as follows.

Sales		Units costs	
Units	Probability	RWF	Probability
10,000	0.8	6	0.7
15,000	0.2	8	0.3

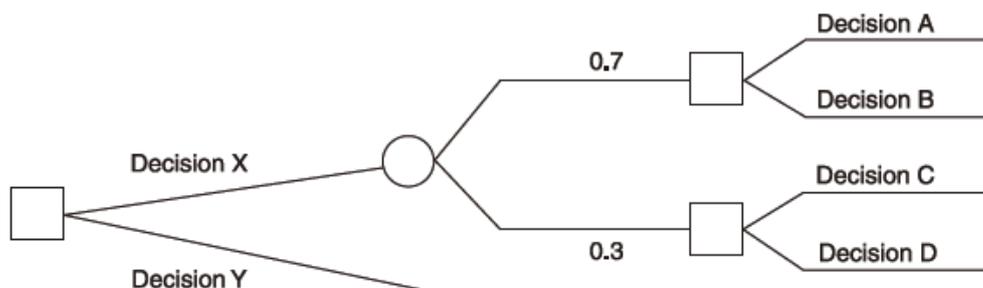
a) The decision tree could be drawn as follows.



b) The layout shown above will usually be easier to use than the alternative way of drawing the tree, which is as follows.



Sometimes, a **decision take now** will lead to **other decisions to be taken in the future**. When this situation arises, the decision tree can be drawn as a **two –stage tree**, as follows.



In this tree, either a choice between A and B or else a choice between C and D will be make, depending on the outcome which occurs after choosing X.

The decision tree should be in **chronological order** from **left to right**. When there are two-stage decision trees, the first decision in time should be drawn on the left.

Example: A decision tree

Beethoven has a new wonder product, the vylin, of which it expects great things. At the moment the company has two courses of action open to it, to test market the product or abandon it.

If the company test markets it, the cost will be RWF100,000 and the market response could be positive or negative with probabilities of 0.060 and 0.40.

If the response is positive the company could either abandon the product or market if full scale.

If it markets the vylin full scale, the outcome might be low, medium or high demand, and the respective net gains/(losses) would be (200) , 200 or 1,000 in units of RWF1,000 (the result could range from a net loss of RWF200,000 to a gain of RWF1,000,000). These outcomes have probabilities of 0.20, 0.50 and 0.30 respectively.

If the result of the test marketing is negative and the company goes ahead and markets the product estimated losses would be RWF600,000.

If, at any point, the company abandon the product, there would be a net gain of RWF50,000 from the sale of scrap. All the financial values have been discounted to the present.

Required

- a) Draw a decision tree
- b) Include figures for cost, loss or profit on the appropriate branches of the tree.

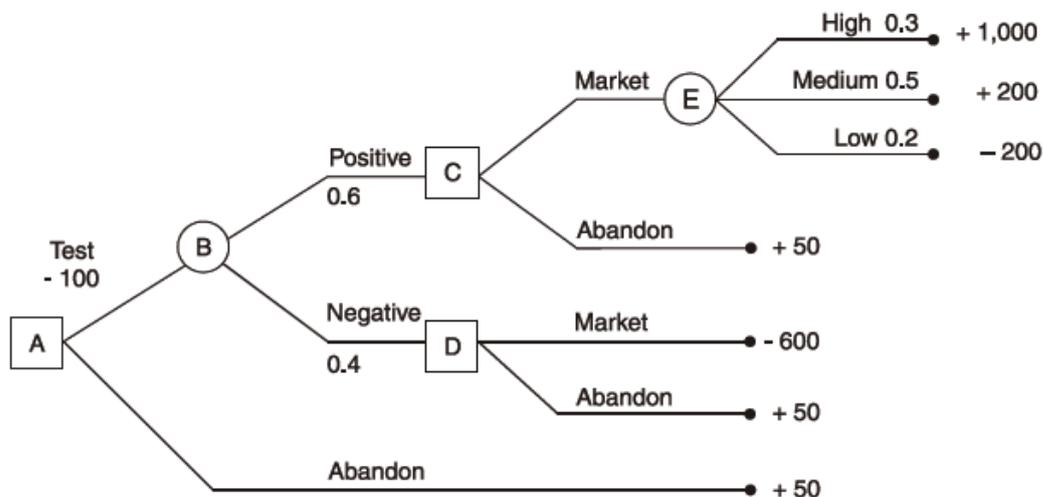
Solution

The starting point for the tree is to establish what decision has to be made now. What are the options?

- a) To test market
- b) To abandon

The outcome of the ‘abandon’ option is known with certainty. There are two possible outcomes of the option to test market, positive response and negative response.

Depending on the outcome of the test marketing, another decision will then be made, to abandon the product or to go ahead.



Evaluating the decision with a decision tree

Rollback analysis evaluates the V of each decision option. You have to work from right to left and calculate EVs at each outcome point.

The EV of each decision option can be evaluated, using the decision tree to help with keeping the logic on track. The basic rules are as follows.

- a) We start on the **right hand side** of the tree and **work back** towards the left hand side and the current decision under consideration . This is sometimes known as the **‘rollback’ technique or ‘rollback analysis’**
- b) Working from **right to left**, we calculate the **EV of revenue**, cost **contribution or profit** at each outcome point on the tree

In the above example, the right-hand-most outcome point is point E, and EV is as follows.

	<i>Profit</i>	<i>Probability</i>	
	<i>x</i>	<i>p</i>	<i>px</i>
	RWF'000		RWF'000
High	1,000	0.3	300
Medium	200	0.5	100
Low	(200)	0.2	<u>(40)</u>
		EV	360

This is the EV of the decision to market the product if the test shows positive response. It may help you to write the EV on the decision tree itself, at the appropriate outcome point (point E).

- a) At **decision point C**, the **choice** is as follows.
 - (i) Market, EV = +360 (the EV at point E)
 - (ii) Abandon, value = + 50

The choice would be to market the product, and so the V at decision point C is +360

b) At **decision point D**, the **choice** is as follows.

(i) Market, value = -600

(ii) Abandon , value =+ 50

The choice would be to abandon, and so the EV at decision point D is +50

The second stage decisions have therefore been made. If the original decision is to test market, the company will market the product if the test shows positive customer response, and will abandon the product if the test results are negative.

The evaluation of the decision tree is completed as follows.

a) **Calculate the EV at outcome point B.**

$$\begin{aligned} & 0.6 \times 360 \quad (\text{Ev at C}) \\ + & 0.4 \times 50 \quad (\text{EV at D}) \\ = & 216 + 20 = 236 \end{aligned}$$

b) **Compare the options at point A**, which are as follows

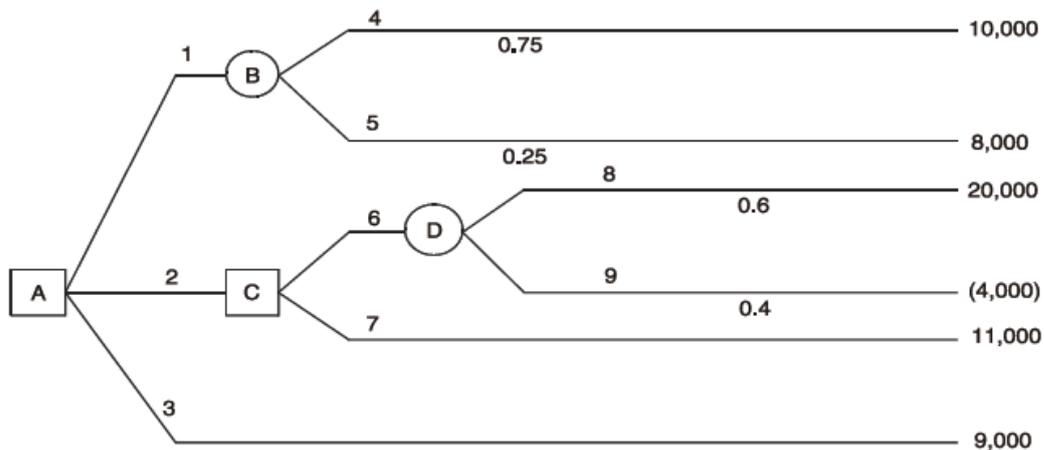
(i) Test: EV =EV at B minus test marketing cost = 236 -100=136

(ii) Abandon: Value = 50

The choice would be to test market the product, because it has a higher **EV of profit**

Question

Consider the following diagram



If a decision maker wished to maximise the value of the outcome, which options should be selected?

- A. Option 2 and option 7
- B. Option 3
- C. Option 1 and option 4
- D. Option 2, option 6 and option 8

Answer

The correct answer is A.

The various outcomes must be evaluated using expected values.

$$\text{EV at point B: } (0.75 \times 10,000) + (0.25 \times 8,000) = 9,500$$

$$\text{EV at point D: } (0.6 \times 20,000) + (0.4 \times (4,000)) = 10,400$$

EV at point C: choice between 10,400 and 11,000

EV at point A : Choice between B (9,500), C (10,400 or 11,000) and choice 3 (9,000).

If we are trying to maximise the figure, option 2 and the option 7 are chosen to give 11,000.

Evaluating decisions by using decision trees has a number of limitations.

- a) The time value of money may not be taken into account.
- b) Decision trees are not very suitable for use in complex situations.
- c) The outcome with the highest EV may have the greatest risks attached to it. Managers may be reluctant to take risks which may lead to losses.
- d) The probabilities associated with different branches of the 'tree' are likely to be estimates, and possibly unreliable or inaccurate.

F. THE VALUE OF INFORMATION

Perfect information is guaranteed to predict the future with 100% accuracy. Imperfect information is better than no information at all but could be wrong in its prediction of the future.

The value of perfect information is the difference between the EV of profit with perfect information and the EV of profit without perfect information.

Perfect information removes all doubt and uncertainty from a decision, and enables managers to make decisions with complete confidence that they have selected the optimum course of action.

The value of perfect information.

Step 1

If we do not have perfect information and we must choose between two or more decision options we would select the decision option which offers the highest EV of profit. This option will not be the best decision under all circumstances. There will be some probability that what was really the best option will not have been selected, given the way actual events turn out.

Step 2

With perfect information, the best decision option will always be selected. The profits from the decision will depend on the future circumstances which are predicted by the information nevertheless, the EV of profit with perfect information should be higher than the EV of profit without the information.

Step 3

The value of perfect information is the difference between these two EVs

Example : the value of perfect information

The management of Ivor Ore must choose whether to go ahead with either of two mutually exclusive projects, A and B. The expected profits are as follows.

	Profit if there is strong demand	Profit/(loss) if there is weak demand
Option A	RWF4,000	RWF(1,000)
Option B	RWF1,500	RWF500
Probability of demand	0.3	0.7

Required

- a) Ascertain what the decision would be, based on expected values, if no information about demand were available.
- b) Calculate the value of perfect information about demand.

Solution

Step 1

If there were no information to help with the decision, the project with the higher EV of profit would be selected.

<i>Probability</i>	<i>Project A</i>		<i>Project B</i>	
	Profit	EV	Profit	EV
	RWF	RWF	RWF	RWF
0.3	4,000	1,200	1,500	450
0.7	(1,000)	<u>(700)</u>	500	350
		<u>500</u>		800

Project B would be selected

This is clearly the better option if demand turns out to be weak. However, if demand were to turn out to be strong, project A would be more profitable. There is a 30% chance that this could happen.

Step 2

Perfect information will indicate for certain whether demand will be weak or strong. If demand is forecast 'weak' project B would be selected. If demand is forecast as 'strong' , project A would be selected, and perfect information would improve the profit from RWF1,500, which would have been earned by selecting B, to RWF4,000

<i>Forecast demand</i>	<i>Probability</i>	<i>Project chosen</i>	<i>Profit RWF</i>	<i>EV of profit RWF</i>
Weak	0.7	B	500	350
Strong	0.3	A	4,000	<u>1,200</u>
EV of profit with perfect information				<u>1,550</u>

Step 3

	RWF
EV of profit without perfect information (ie if project B is always chosen)	800
EV of profit with perfect information	<u>1,550</u>
	<u>750</u>

Provided that the information does not cost more than RWF750 to collect, it would be worth having.

Question

WL must decide at what level to market a new product, the urk. The urk can be sold nationally, within a single sales region (where demand is likely to be relatively strong) or within a single area. The decision is complicated by uncertainty about the general strength of

consumer demand for the product, and the following conditional profit table has been constructed.

		<i>Weak</i>	<i>Demand Moderate</i>	<i>Strong</i>
		<i>RWF</i>	<i>RWF</i>	<i>RWF</i>
Market	Nationally (A)	(4,000)	2,000	10,000
	In one region (b)	0	3,500	4,000
	In one area (C)	1,000	1,500	2,000
Probability		0.3	0.5	0.2

Required

Option B should be selected, based on EVs of profit. True or False?

Answer

The correct answer is option B and so the statement is true.

Without perfect information, the option with the highest EV of profit will be chosen.

<i>Probability</i>	<i>Option A Profit</i>	<i>(National) EV</i>	<i>Option B Profit</i>	<i>(Regional) EV</i>	<i>Option C Profit</i>	<i>EV</i>
	<i>RWF</i>	<i>RWF</i>	<i>RWF</i>	<i>RWF</i>	<i>RWF</i>	<i>RWF</i>
0.3	(4,000)	(1,200)	0	0	1,000	300
0.5	2,000	1,000	3,500	1,750	1,500	750
0.2	10,000	<u>2,000</u>	4,000	<u>800</u>	2,000	<u>400</u>
		1,800		2,550		1,450

Marketing regionally (option B) has the highest EV of profit, and would be selected.

Question

Use the information in your answer to the question above (Decision based on EV of profit)

Required

Calculate the value of perfect information about the state of demand.

Answer

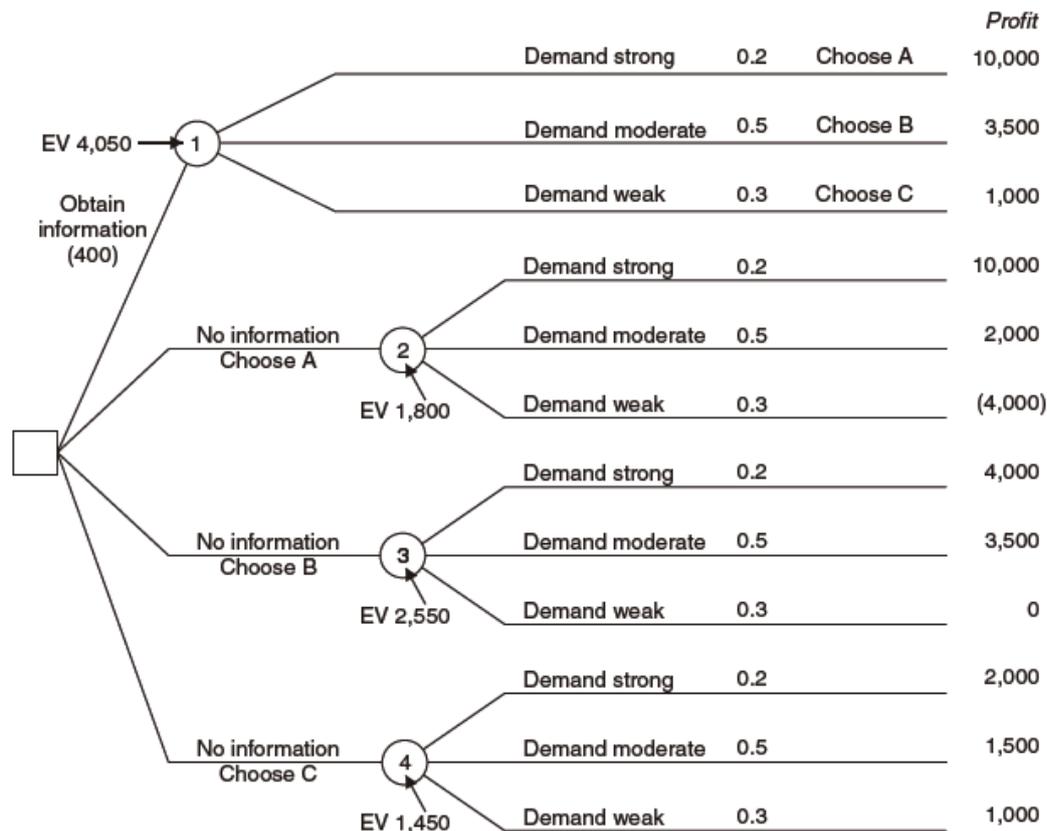
The correct answer is RWF1,500.

If perfect information about the state of consumer demand were available, option A would be preferred if the forecast demand is strong and option C would be preferred if the forecast demand is weak.

	<i>Probability</i>	<i>Choice</i>	<i>Profit</i> RWF	<i>EV of profit</i> RWF
Weak	0.3	C	1,000	300
Moderate	0.5	B	3,500	1,750
Strong	0.2	A	10,000	<u>2,000</u>
EV of profit with perfect information				4,050
EV of profit, selecting option B				<u>2,550</u>
Value of perfect information				<u>1,500</u>

Perfect information and decision trees

When the option exists to obtain information, the decision can be shown, like any other decision, in the form of a decision tree, as follows. We will suppose, for illustration, that the cost of obtaining perfect information is RWF400.



The

decision would be to obtain perfect information, since the EV of profit is RWF4,050 - RWF400 = RWF3,650.

You should check carefully that you understand the logic of this decision and that you can identify how the EVs at outcome boxes 1, 2, 3 and 4 have been calculated.

The value of imperfect information

There is one serious drawback to the technique we have just looked at: in practice, useful information is never perfect unless the person providing it is the sole source of the uncertainty. Market research findings or information from pilot tests and so on are likely to be reasonably accurate, but they can still be wrong: they provide imperfect information. It is possible, however, to arrive at an assessment of **how much it would be worth paying for**

such imperfect information, given that we have a rough indication of how right or wrong it is likely to be.

Suppose we are considering the sex and hair colour of people in a given group or population consisting of 70% men and 30% women. We have established the probabilities of hair colourings as follows:

	Men	Women
Brown	0.60	0.35
Blonde	0.35	0.55
Red	0.05	0.10

This shows, for example, that 5% of men in such a sample have red hair. These probabilities of sex and hair colouring might be referred to as prior probabilities.

Posterior probabilities consider the situation in reverse or retrospect, so that we can ask the question: ‘Given that a person taken at random from the population is brown-haired what is the probability that the person is male (or female)?’

The information can be presented in a table. Let’s suppose that the population consists of 1,000 people.

	Male	Female	Total
Brown	420 (W3)	105 (W4)	525 (W5)
Blonde	245	165	410
Red	<u>35</u>	<u>30</u>	<u>65</u>
	<u>700</u> (W1)	<u>300</u> (W2)	<u>1,000</u>

Workings

- 1 $1,000 \times 70\%$
- 2 $1,000 - 700$
- 3 $700 \times 60\%$ (the other two values in the column being calculated in a similar way)

4 $300 \times 35\%$ (the other two values in the column being calculated in a similar way)

5 $420 + 105$ (the other two values in the column being calculated in a similar way)

$\therefore P(\text{Person selected is a male, given that that person is brown-haired}) = 420/525 = 0.8$

Example: The value of imperfect information

Suppose that the Small Oil Company (SOC) is trying to decide whether or not to drill on a particular site. The chief engineer has assessed the probability that there will be oil, based on vast experience, as 20% and the probability that there won't be oil as 80%.

It is possible for the SOC to hire a firm of international consultants to carry out a complete survey of the site. SOC has used the firm many times before and has estimated that if there really is oil, there is a 95% chance that the report will be favourable, but if there is no oil, there is only a 10% chance that the report will indicate there is oil.

Required

Determine whether drilling should occur.

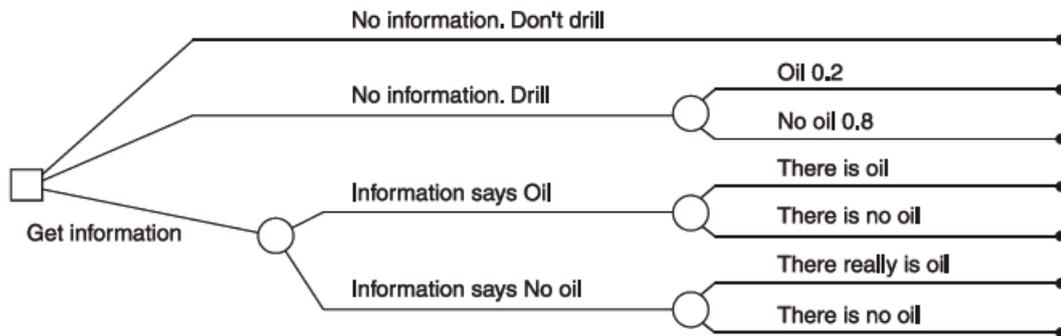
Solution

Read the information given carefully. We are given three sets of probabilities.

- a) The probability that there will be oil (0.2) or there will not be (0.8). These outcomes are mutually exclusive.
- b) The probability that, If there is oil, the report will say there is oil (0.95) or say there is no oil (0.05)
- c) The probability that, if there is no oil, the report will say there is oil (0.1) or say there is no oil (0.9).

Both (b) and (c) describe conditional events, since the existence of oil or otherwise influences the chances of the survey report being correct.

SOC, meanwhile faces a number of choices which we can show as a decision tree.



We must now calculate the probabilities of the following outcomes.

- The information will say 'oil' or 'no oil'
- The information will be right or wrong if it says 'oil'
- The information will be right or wrong if it says 'no oil'

If you check the information given in the problem, you will find that these probabilities are not given.

- a) We are told that the engineer has assessed that there is a 20% chance of oil and an 80% chance of no oil (ignoring information entirely). These are the **prior probabilities** of future possible outcomes.
- b) The **probabilities that there will be oil or no oil once the information has been obtained are posterior probabilities.**

Step 1

We can tabulate the various probabilities as percentages.

		Oil		No Oil		Total	
Survey	Oil	19	(w2)	8	(w3)	27	(w4)
Result:	No oil	<u>1</u>		<u>72</u>		<u>73</u>	
Total		<u>20</u>	(w1)	<u>80</u>		<u>100</u>	

Workings

1. The engineer estimates 20% probability of oil and 80% of no oil.
2. If there is oil, i.e. in 20 cases out of 100, the survey will say so in 95% of these cases, i.e. in $20 \times 0.95 = 19$ cases. The 1 below the 19 is obtained by subtraction.
3. In the 80 per 100 cases where there is in fact no oil, the survey will wrongly say that there is oil 10% of the time; i.e. $80 \times 0.10 = 8$ cases. The 72 below the 8 is obtained by subtraction.
4. The horizontal totals are given by addition.

Step 2

We can now provide all the probabilities needed to complete the tree.

$$P(\text{survey will say there is oil}) = 27/100 = 0.27$$

$$P(\text{survey will say there is no oil}) = 73/100 = 0.73$$

$$\text{If survey says oil } P(\text{there is oil}) = 19/27 = 0.704$$

$$P(\text{there is no oil}) = 8/27 = 0.296 \text{ (or } 1 - 0.704)$$

$$\text{If survey says no oil } P(\text{there is oil}) = 1/73 = 0.014$$

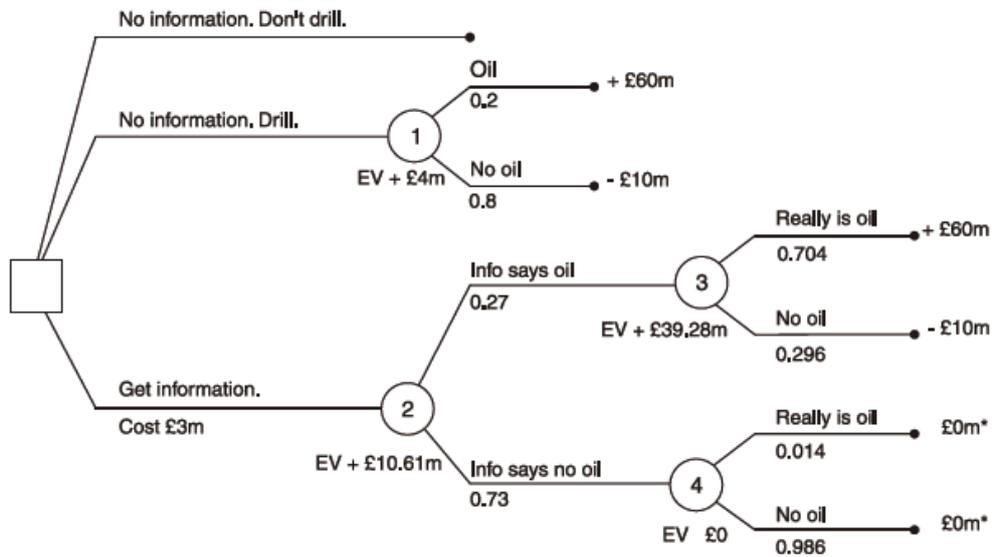
$$P(\text{there is no oil}) = 72/73 = 0.986 \text{ (or } 1 - 0.014)$$

Step 3

We can now go on to complete the decision tree. Let us make the following assumptions.

- The cost of drilling is RWF10m.
- The value of the benefits if oil is found is RWF70m, giving a net 'profit' of RWF60m
- The cost of obtaining information from the consultants would be RWF3m.

An assumption is made that the decision maker will take whichever decision the information indicates is the best. If the information says 'oil', the company will drill and if the information says 'no oil' it will not drill.



The information is 'no oil' @, so the company won't drill, regardless of whether there really is oil or not.

Step 4

We can now perform rollback analysis.

		RWFm
EV at point 2 =	$0.704 \times \text{RWF}60\text{m}$	42.24
	$0.296 \times (\text{RWF}10\text{m})$	(2.96)
		+39.28

		RWFm
EV at point 2 =	0.27xRWF39.28m	10.61
	0.73xRWF0	0.00
		+10.61

Step 5

There are three choices	EV
a) Do not obtain information and do not drill	RWFO
b) Do not obtain information and drill	+RWF4million
c) Obtain information first, decide about drilling later (RWF(10.61m – 3m))	+7.61million

The decision should be to obtain the information from a survey first.

Step 6

The value of the imperfect information is the difference between (b) and (c) , RWF3.61 million.

G. SENSITIVITY ANALYSIS

Sensitivity analysis can be used in any situation so long as the relationships between the key variables can be established. Typically this involves changing the value of a variable and seeing how the results are affected.

Approaches to sensitivity analysis

Sensitivity analysis is a term used to describe any technique whereby decision options are tested for their vulnerability to changes in any 'variable' such as expected sales volume, sales price per unit, material costs, or labour costs.

Here are three useful approaches to sensitivity analysis.

- a) To estimate by **how much costs and revenues would need to differ** from their estimated values before the decision would change.
- b) To estimate whether a decision would change if estimated costs were **x% higher** than estimated, or estimated revenues **y% lower** than estimated.
- c) To estimate by how much costs and/or revenues would need to differ from their estimated values before the decision maker would be **indifferent** between two options.

The essence of the approach, therefore, is to carry out the calculations with one set of values for the variables and then substitute other possible values for the variables to see how this affects the overall outcome.

- a) From your studies of information technology you may recognise this as what if analysis that can be carried out using a spreadsheet.
- b) From your studies of linear programming you may remember that sensitivity analysis can be carried out to determine over which ranges the various constraints have an impact on the optimum solution.
- c) Flexible budgeting can also be a form of sensitivity analysis.

Example: sensitivity analysis

Sensivite has estimated the following sales and profits for a new product which it may launch on to the market.

		RWF	RWF
Sales	(2,000 units)		4,000
Variable costs:	materials	2,000	
	labour	<u>1,000</u>	
			<u>3,000</u>
Contribution			1,000
Less incremental fixed costs			<u>800</u>
Profit			<u><u>200</u></u>

Required

Analyse the sensitivity of the project.

Solution

- If incremental **fixed costs** are more than 25% above estimate, the project would make a loss.
- If **unit costs of materials** are more than 10% above estimate, the project would make a loss.
- Similarly, the project would be sensitive to an increase in unit labour costs of more than RWF200, which is 20% above estimate, or else to a drop in the **unit selling price** of more than 5%.
- The **margin of safety**, given a breakeven point of 1,600 units, is $(400/2,000) \times 100\% = 20\%$.

Management would then be able to judge more clearly whether the product is likely to be profitable. The items to which profitability is most sensitive in this example are the selling price (5%) and material costs (10%). Sensitivity analysis can help to **concentrate management attention** on the most important factors.

H. SIMULATION MODELS

Simulation models can be used to deal with decision problems involving a number of uncertain variables. **Random numbers** are used to assign values to the variables.

One of the chief problems encountered in decision making is the uncertainty of the future. Where only a few factors are involved, probability analysis and expected value calculations can be used to find the most likely outcome of a decision. Often, however, in real life, there are so **many uncertain variables** that this approach does not give a true impression of possible variations in outcome.

To get an idea of what will happen in real life one possibility is to use a **simulation model** in which the **values and the variables are selected at random**. Obviously this is a situation **ideally suited to a computer** (large volume of data, random number generation).

The term 'simulation' model is often used more specifically to refer to modelling which **makes use of random numbers**. This is the '**Monte Carlo**' method of simulation. In the business environment it can, for example, be used to examine inventory, queuing, scheduling and forecasting problems.

Random numbers are allocated to each possible value of the uncertain variable in proportion to the probabilities, so that a probability of 0.1 gets 10% of the total numbers to be assigned. These random numbers are used to assign values to the variables.

Example: simulation and spreadsheets

A supermarket sells a product for which the daily demand varies. An analysis of daily demand over a period of about a year shows the following probability distribution.

Demand per day	Probability
<i>Units</i>	
35	0.10
36	0.20
37	0.25
38	0.30
39	0.08
40	<u>0.07</u>
	<u>1.00</u>

To develop a simulation model in which one of the variables is daily demand, we would **assign a group of numbers to each value for daily demand**. The probabilities are stated to two decimal places, and so there must be 100 random numbers in total, 00 – 99 (we use 00-99 rather than 1-100 so that we can use two-digit random numbers.) Random numbers are assigned in proportion to the **probabilities**, so that a probability of 0.1 gets 10% of the total numbers to be assigned, that is 10 numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

The assignments would therefore be as follows.

Demand per day	Probability	Numbers assigned
<i>Units</i>		
35	0.10	00 – 09
36	0.20	10 – 29
37	0.25	30 – 54
38	0.30	55 – 84
39	0.08	85 – 92
40	0.07	93 – 99

When the simulation model is run, random numbers will be generated to derive values for daily demand. For example, if the model is used to simulate demand over a ten day period, the random numbers generated might be as follows.

19007174604721296802

The model would then **assign values** to the demand per day as follows.

Day	Random number	Demand
<i>Units</i>		
1	19	36
2	00	35
3	71	38
4	74	38
5	60	38
6	47	37
7	21	36
8	29	36
9	68	38
10	02	35

You might notice that on none of the ten days is the demand 39 or 40 units, because the random numbers generated did not include any value in the range 85 – 99. When a simulation model is used, there must be a long enough run to give a good representation of the system and all its potential variations.

Uses of simulation

In the supermarket example above, the supermarket would use the information to minimise inventory holding without risking running out of the product. This will reduce costs but avoid lost sales and profit.

A supermarket can also use this technique to estimate queues with predicted length of waiting time determining the number of staff required.

STATISTICAL FORMULAE

1. The Arithmetic mean (A.M.)

$$A.M. = \frac{\sum fx}{\sum f}$$

or

$$A.M. = A + \frac{\sum fd}{\sum f}$$

where A = assumed mean
 d = deviation of class mark from A

2. The standard deviation (S.D.)

where $x = X - \bar{X}$.

$$S.D. = \sqrt{\frac{\sum fx^2}{\sum f}}$$

$$\text{or } S.D. = \sqrt{\frac{\sum fd^2}{\sum f} - \left\{ \frac{\sum fd}{\sum f} \right\}^2}$$

3. The Binomial distribution

$$\text{mean} = np$$

$$S.D. = \sqrt{npq}$$

The probability of x successes in a sample of n events, where p is the probability of a success on a single trial, is given by

$$P_x = \frac{n!}{(x!)(n-x)!} p^x q^{n-x}$$

$$P_{x+1} = \frac{n-x}{x+1} \frac{p}{q} \cdot P_x$$

4. The Poisson distribution

$$\text{mean} = m = np$$

$$S.D. = \sqrt{m}$$

The probability of x successes

$$P_x = \frac{m^x e^{-m}}{x!} = \frac{m^x}{x!} \frac{1}{(2.718)^m}$$

$$P_{x+1} = \frac{m}{x+1} \cdot P_x$$

5. Standard errors (S.E.)

$$\begin{aligned}
 S.E. \text{ of mean} &= \frac{\sigma}{\sqrt{n}} \\
 S.E. \text{ of difference of means} &= \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \\
 S.E. \text{ of proportion} &= \sqrt{\frac{pq}{n}} \\
 S.E. \text{ of differences of proportions} &= \sqrt{\frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}}
 \end{aligned}$$

6. Index Numbers

Price Indices

$$\text{Laspeyres} = \frac{\sum P_n Q_o}{\sum P_o Q_o} \times 100$$

$$\text{Paasche} = \frac{\sum P_n Q_n}{\sum P_o Q_n} \times 100$$

Quantity Indices

$$\text{Laspeyres} = \frac{\sum Q_n P_o}{\sum Q_o P_o} \times 100$$

$$\text{Paasche} = \frac{\sum Q_n P_n}{\sum Q_o P_n} \times 100$$

7. Regression lines

To determine the regression line $Y = a + bX$. Use

$$\begin{aligned}
 \sum Y &= na + b\sum X \\
 \sum XY &= a\sum X + b\sum X^2
 \end{aligned}$$

Alternative method of calculating the regression equation

$$\begin{aligned}
 Y &= a + bx \\
 b &= \frac{\sum xy - \frac{\sum x \times \sum y}{n}}{\sum x^2 - \frac{(\sum x)^2}{n}} \\
 a &= \frac{\sum y}{n} - \frac{b\sum x}{n}
 \end{aligned}$$

8. Correlation co-efficients

(a) Product moment

$$r = \frac{\sum xy}{\sqrt{\sum x^2 \sum y^2}}$$

where

$$x = (X - \bar{X})$$

$$y = (Y - \bar{Y})$$

or

$$r = \frac{\text{co-variance}}{S.D._x S.D._y}$$

or

$$r = \frac{\frac{\sum XY}{N} - \frac{\sum X}{N} \times \frac{\sum Y}{N}}{\sqrt{\left[\frac{\sum x^2}{N} - \left(\frac{\sum x}{N} \right)^2 \right] \left[\frac{\sum y^2}{N} - \left(\frac{\sum y}{N} \right)^2 \right]}}$$

(b) Rank

$$(R) = 1 - \frac{6 \sum (\text{differences})^2}{N(N^2 - 1)}$$

TABLE 2

PRESENT VALUE OF €1.00

$$P = \frac{S}{(1 + o)^n}$$

PERIODS	4%	6%	8%	10%	12%	14%	16%	18%	20%	22%	24%	26%	28%	30%	40%
0.652	0.847	0.833	0.820	0.806	0.794	0.781	0.769	0.714	1	0.962	0.943	0.926	0.909	0.893	0.877
0.743	0.718	0.694	0.672	0.650	0.630	0.610	0.592	0.510	2	0.925	0.890	0.857	0.826	0.797	0.769
0.641	0.609	0.579	0.551	0.524	0.500	0.477	0.455	0.364	3	0.889	0.840	0.794	0.751	0.712	0.675
0.552	0.516	0.482	0.451	0.423	0.397	0.373	0.350	0.260	4	0.855	0.792	0.735	0.683	0.636	0.592
0.476	0.437	0.402	0.370	0.341	0.315	0.291	0.269	0.186	5	0.822	0.747	0.681	0.621	0.567	0.519
0.410	0.370	0.335	0.303	0.275	0.250	0.227	0.207	0.133	6	0.790	0.705	0.630	0.564	0.507	0.456
0.354	0.314	0.279	0.249	0.222	0.198	0.178	0.159	0.095	7	0.760	0.665	0.583	0.513	0.452	0.400
0.305	0.266	0.233	0.204	0.179	0.157	0.139	0.123	0.068	8	0.731	0.627	0.540	0.467	0.404	0.351
0.263	0.225	0.194	0.167	0.144	0.125	0.108	0.094	0.048	9	0.703	0.592	0.500	0.424	0.361	0.308
0.227	0.191	0.162	0.137	0.116	0.099	0.085	0.073	0.035	10	0.676	0.558	0.463	0.385	0.322	0.270
0.195	0.162	0.135	0.112	0.094	0.079	0.066	0.056	0.025	11	0.650	0.527	0.429	0.350	0.287	0.237
0.168	0.137	0.112	0.092	0.076	0.062	0.052	0.043	0.018	12	0.625	0.497	0.397	0.319	0.257	0.208
0.145	0.116	0.093	0.075	0.061	0.050	0.040	0.033	0.013	13	0.601	0.469	0.368	0.290	0.229	0.182
0.125	0.099	0.078	0.062	0.049	0.039	0.032	0.025	0.009	14	0.577	0.442	0.340	0.263	0.205	0.160
0.108	0.084	0.065	0.051	0.040	0.031	0.025	0.020	0.006	15	0.555	0.417	0.315	0.239	0.183	0.140
0.093	0.071	0.054	0.042	0.032	0.025	0.019	0.015	0.005	16	0.534	0.394	0.292	0.218	0.163	0.123
0.080	0.060	0.045	0.034	0.026	0.020	0.015	0.012	0.003	17	0.513	0.371	0.270	0.198	0.146	0.108
0.069	0.051	0.038	0.028	0.021	0.016	0.012	0.009	0.002	18	0.494	0.350	0.250	0.180	0.130	0.095
0.060	0.043	0.031	0.023	0.017	0.012	0.009	0.007	0.002	19	0.475	0.331	0.232	0.164	0.116	0.083
0.051	0.037	0.026	0.019	0.014	0.010	0.007	0.005	0.001	20	0.456	0.312	0.215	0.149	0.104	0.073
0.044	0.031	0.022	0.015	0.011	0.008	0.006	0.004	0.001	21	0.439	0.294	0.199	0.135	0.093	0.064
0.038	0.026	0.018	0.013	0.009	0.006	0.004	0.003	0.001	22	0.422	0.278	0.184	0.123	0.083	0.056
0.033	0.022	0.015	0.010	0.007	0.005	0.003	0.002		23	0.406	0.262	0.170	0.112	0.074	0.049
0.028	0.019	0.013	0.008	0.006	0.004	0.003	0.002		24	0.390	0.247	0.158	0.102	0.066	0.043
0.024	0.016	0.010	0.007	0.005	0.003	0.002	0.001		25	0.375	0.233	0.146	0.092	0.059	0.038
0.021	0.014	0.009	0.006	0.004	0.002	0.002	0.001		26	0.361	0.220	0.135	0.084	0.053	0.033
0.018	0.011	0.007	0.005	0.003	0.002	0.001	0.001		27	0.347	0.207	0.125	0.076	0.047	0.029
0.016	0.010	0.006	0.004	0.002	0.002	0.001	0.001		28	0.333	0.196	0.116	0.069	0.042	0.026
0.014	0.008	0.005	0.003	0.002	0.001	0.001	0.001		29	0.321	0.185	0.107	0.063	0.037	0.022
0.012	0.007	0.004	0.003	0.002	0.001	0.001	0.001		30	0.308	0.174	0.099	0.057	0.033	0.020
0.003	0.001	0.001	0.001	0.001	0.001	0.001	0.001		40	0.208	0.097	0.046	0.022	0.011	0.005

PRESENT VALUE ANNUITY FACTORS: PRESENT VALUE OF €1
RECEIVED ANNUALLY FOR n YEARS $\left(\frac{1 - (1 + r)^{-n}}{r}\right)$

DISCOUNT RATES (r)%

Years (n)	1%	2%	4%	6%	8%	10%	12%	14%	15%	16%	18%	20%	22%	24%	25%	26%	28%	30%
1	0.990	0.980	0.962	0.943	0.926	0.909	0.893	0.877	0.870	0.862	0.847	0.833	0.820	0.806	0.800	0.794	0.781	0.769
2	1.970	1.942	1.886	1.833	1.783	1.736	1.690	1.647	1.626	1.605	1.566	1.528	1.492	1.457	1.440	1.424	1.392	1.361
3	2.941	2.884	2.775	2.675	2.577	2.487	2.402	2.322	2.283	2.246	2.174	2.106	2.042	1.981	1.952	1.923	1.868	1.816
4	3.902	3.808	3.610	3.465	3.312	3.170	3.037	2.914	2.855	2.798	2.690	2.589	2.494	2.404	2.362	2.320	2.241	2.166
5	4.853	4.713	4.452	4.212	3.996	3.791	3.605	3.433	3.352	3.274	3.127	2.991	2.864	2.745	2.689	2.635	2.532	2.436
6	5.795	5.601	5.242	4.917	4.623	4.355	4.111	3.889	3.784	3.685	3.498	3.326	3.167	3.020	2.951	2.885	2.759	2.643
7	6.728	6.472	6.002	5.582	5.206	4.868	4.564	4.288	4.160	4.039	3.812	3.605	3.416	3.242	3.161	3.083	2.937	2.802
8	7.652	7.325	6.733	6.210	5.747	5.335	4.968	4.639	4.487	4.344	4.078	3.837	3.619	3.421	3.329	3.241	3.076	2.925
9	8.566	8.162	7.435	6.802	6.247	5.759	5.328	4.946	4.772	4.607	4.303	4.031	3.786	3.566	3.463	3.366	3.184	3.019
10	9.471	8.983	8.111	7.360	6.710	6.145	5.650	5.216	5.019	4.833	4.494	4.192	3.923	3.682	3.571	3.465	3.269	3.092
11	10.368	9.787	8.760	7.887	7.139	6.495	5.988	5.453	5.234	5.029	4.636	4.327	4.035	3.766	3.656	3.544	3.335	3.147
12	11.255	10.575	9.385	8.384	7.536	6.814	6.194	5.660	5.421	5.197	4.793	4.439	4.127	3.851	3.725	3.606	3.387	3.190
13	12.114	11.343	9.986	8.853	7.904	7.103	6.424	5.842	5.583	5.342	4.910	4.533	4.203	3.912	3.780	3.656	3.427	3.223
14	13.004	12.106	10.563	9.295	8.244	7.367	6.628	6.002	5.724	5.468	5.008	4.611	4.265	3.961	3.824	3.695	3.459	3.249
15	13.865	12.849	11.118	9.712	8.559	7.606	6.811	6.142	5.847	5.575	5.092	4.675	4.315	4.001	3.859	3.726	3.483	3.268
16	14.718	13.578	11.652	10.106	8.851	7.824	6.974	6.265	5.954	5.669	5.162	4.730	4.357	4.033	3.887	3.751	3.503	3.283
17	15.562	14.292	12.166	10.477	9.122	8.022	7.120	6.373	6.047	5.749	5.222	4.775	4.391	4.059	3.910	3.771	3.518	3.295
18	16.328	14.992	12.659	10.828	9.372	8.201	7.250	6.467	6.128	5.818	5.273	4.812	4.419	4.080	3.928	3.786	3.529	3.304
19	17.226	15.678	13.134	11.158	9.604	8.365	7.366	6.550	6.198	5.877	5.316	4.844	4.442	4.097	3.942	3.799	3.539	3.311
20	18.046	16.351	13.590	11.470	9.818	8.514	7.469	6.623	6.259	5.929	5.353	4.870	4.460	4.110	3.954	3.808	3.546	3.316
21	18.857	17.011	14.029	11.764	10.017	8.649	7.562	6.687	6.312	5.973	5.384	4.891	4.476	4.121	3.963	3.816	3.551	3.320
22	19.660	17.658	14.451	12.042	10.201	8.772	7.645	6.743	6.369	6.011	5.410	4.909	4.488	4.130	3.970	3.822	3.556	3.323
23	20.456	18.292	14.857	12.303	10.371	8.883	7.718	6.792	6.399	6.044	5.432	4.925	4.499	4.137	3.976	3.827	3.559	3.325
24	21.243	18.914	15.247	12.550	10.529	8.985	7.784	6.815	6.434	6.073	5.451	4.937	4.507	4.143	3.981	3.831	3.562	3.327
25	22.023	19.523	15.622	12.783	10.675	9.077	7.843	6.873	6.464	6.097	5.467	4.948	4.514	4.147	3.985	3.834	3.564	3.329